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AN UNDERGRADUATE EXERCISE IN MANIPULATION

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Abstract

This paper contains an example of a problem which undergraduates (or even graduates) may care to tackle to improve their manipulative skills. It is quite a taxing problem, whose solution could profitably be spread over several weeks' work. It may be read in conjunction with my article 'Room to wriggle' (Hammersley (1988)).

PROBLEM SOLVING; SKILL AT MATHEMATICAL MANIPULATION

Speaking of his book *How To Solve It*, Pólya said: 'Solving problems is a practical art, like swimming, or skiing, or playing the piano; you can learn it only by imitation or practice.... Our knowledge about any subject consists of *information* and *know-how*. If you have genuine *bona fide* experience of mathematical work on any level, elementary or advanced, there will be no doubt in your mind that, in mathematics, know-how is much more important than mere possession of information. Therefore, in the high school, as on any other level, we should impart, along with a certain amount of information, a certain degree of *know-how* to the student. What is know-how in mathematics? The ability to solve problems—not merely routine problems but problems requiring some degree of independence, judgment, originality, creativity. Therefore the first and foremost duty of the high school in teaching mathematics is to emphasize *methodical work in problem solving*. That is my conviction....'

Undergraduate courses in mathematics do not pay enough attention to Pólya's dicta, largely because the syllabus is overloaded with lectures devoted to purveying information, and too often examination questions expect little more than regurgitation of lecture notes. Even when riders are set, their manipulative and problem-solving content has to be sufficiently slight to be handled within the time constraints of a three-hour examination. To develop manipulative skills, one needs to wrestle with a problem, which can be a time-consuming business. You have to fiddle with expressions and equations to get them into amenable shape. Examination questions often indicate to candidates the particular technique (familarly signalled by the phrase 'hence or otherwise') that will most expeditiously produce an answer. This spoonfeeding is a serious educational shortcoming of examinations, because undergraduates do not have to pinpoint *for themselves* that bit of their store of lecture-derived information that will

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serve their immediate purpose: they do not learn what is useful for what, and do not acquire the habits of *judgment* and *independence* of which Pólya speaks.

In this paper I shall give an example of a single problem which I hope may serve as an exercise for undergraduates wishing to develop their manipulative skills. It does not require any technical knowledge outside the ordinary undergraduate syllabus, but it is not a particularly easy problem. It calls for perseverance and it would not serve its purpose if it were easy enough for the average undergraduate to dispose of in an hour or two. The paper contains a solution of the problem and also some hints and suggestions for the benefit of those who may get stuck at various intermediate stages of the solution. But it is important that you should manage as much as you possibly can from your own resources. So read the paper very slowly, resisting all temptations to look ahead before you have made determined efforts to manage on your own. A few of the hints and suggestions may be incomplete or even misleading, in the deliberate intent that you should exercise some judgment in problem-solving on whether this or that course of action is likely to be profitable.

To state the problem, consider the sequence of polynomials

$$p_n(y) = \sum_{r=0}^n a_{nr} y^r \quad (1)$$

generated by the recurrence relation

$$p_n(y) = (y - \tfrac{1}{2})^n - \sum_{m=0}^{n-1} \binom{2n}{2m} (-\tfrac{1}{2})^{n-m} p_m(y) \quad (n = 1, 2, \dots) \quad (2)$$

starting from $p_0(y) = 1$. Prove or disprove the conjecture that the coefficients a_{nr} ($1 \leq r \leq n$) are all positive integers, find formulae for these coefficients, and in particular an asymptotic formula when $n \rightarrow \infty$. Find an estimate for $a_{1000, 500}$. Well, good hunting; see how far you can get on your own before reading anything further.

In the meantime, partly to prevent your eye straying towards the first suggestion, let me say something about the background to this particular problem. It is not an artificially created exercise: it arose naturally from a question set in the Final Honours Examination in Mathematics at Oxford in 1985. Candidates had to prove that a non-linear second-order differential equation had a unique solution under certain conditions, and to find bounds for this solution. They had attended lectures on functional analysis; so the question was dressed up in the language of Banach spaces and could be answered in such language to the satisfaction of the examiners. However, functional analysis is a general-purpose technique which is apt to banish interesting problems with a mere wave of Franco-Polish gesticulation. In particular, the use of a supremum norm in a Banach space is liable to waste pertinent information and hence to yield unnecessarily weak results. By ridding this particular examination question of its jargon and recasting it in terms of elementary classical analysis, I was able to prove that the solution of the differential equation existed and was unique under much more general conditions than the examination question had stipulated; and I was able to find much more precise bounds for the solution, indeed

bounds which were everywhere best-possible. Of course, this involved more detailed work than could have been expected under examination conditions; but it does illustrate the extent to which too much information, especially of the soft mathematical species purveyed in lectures, has come to dominate and exclude the fostering of know-how in undergraduate studies. In the course of this investigation, I ran across the polynomials defined by (2). I suspect that they are well known to experts in differential equations; but personally I had not met them before, so I had to start from scratch and work out their properties on my own. They have some features in common with the Bernoulli polynomials; but the Bernoulli polynomials have both positive and negative coefficients which are not all integers: for example, the eighth Bernoulli polynomial is

$$\frac{2}{3}y^2 - \frac{7}{3}y^4 + \frac{14}{3}y^6 - 4y^7 + y^8$$

when normalized to have unit coefficient for its highest power of y . There is little in the definition (2) to suggest that the coefficients in $p_n(y)$ might not be positive and negative rational fractions. So I was surprised to discover on working out the first 5 of the p_n that a_{nr} ($1 \leq r \leq n \leq 5$) are all positive integers. Could this also be true without the restriction $n \leq 5$ and could I find formulae for these coefficients?

Suggestion 1. Use (2) to calculate $p_n(y)$ for $1 \leq n \leq 5$ and examine the behaviour of the coefficients. Prove that $a_{nn} = 1$. Carry the calculation beyond $n = 5$ as far as you can on a computer. Find a formula for $a_{n,n-1}$ for $n \geq 2$.

Amongst the software found in computing laboratories nowadays there are general-purpose packages for manipulating polynomials and other algebraic formulae. For example, with the package MACSYMA, one can feed in (2) more or less as it stands: specifically in the form

```
p_n: if n = 0 then 1
      else (y - 1/4)^n - sum((1/4)^(n-m) binomial(2n, 2m) p_m, m, 0, n-1)
ratsimp.
```

The command `ratsimp` caused the machine (in this case a mainframe VAX 785) to get on with the business of making substitutions and finally printing out the polynomials p_n with the coefficients expressed as rational fractions in their lowest terms. The great advantage of this for the lazy man is that he does not have to think: all the thinking has already been done for him by those who originally compiled the package. On the other hand the sacrifice made for the sake of laziness is rather similar to the sacrifice made in using functional analysis: general-purpose tools are less efficient and necessarily more cumbersome than tools specifically designed for particular situations. A general-purpose package can be rather slow and use a great deal of storage space. Here the machine successfully got as far as $n = 12$, taking 5 minutes for the job; but it gave up the ghost at $n = 13$ because it had used up all its allocated storage space. Nevertheless, 5 minutes machine time for $n \leq 12$ is a great improvement on 15 minutes work without a machine for $n \leq 5$. Here are the results of the hand calculations for $n \leq 5$ and (in Table 1) the machine results for $6 \leq n \leq 12$:

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$p_1 = y$
 $p_2 = y + y^2$
 $p_3 = 3y + 3y^2 + y^3$
 $p_4 = 17y + 17y^2 + 6y^3 + y^4$
 $p_5 = 155y + 155y^2 + 55y^3 + 10y^4 + y^5$

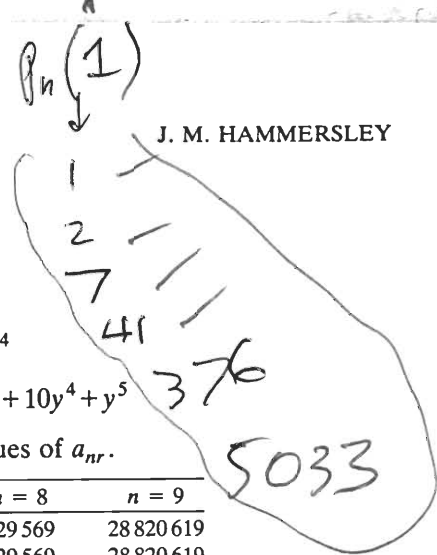


Table 1. Specimen values of a_{nr} .

r	$n = 6$	$n = 7$	$n = 8$	$n = 9$
1	2073	38227	929569	28820619
2	2073	38227	929569	28820619
3	736	13573	330058	10233219
4	135	2492	60605	1879038
5	15	280	6818	211419
6	1	21	518	16086
7		1	28	882
8			1	36
9				1

r	$n = 10$	$n = 11$	$n = 12$
1	1109652905	51943281731	2905151042481
2	1109652905	51943281731	2905151042481
3	393999940	18443289777	1031520165710
4	72346915	3386587600	189409454649
5	8140201	381046666	21311675172
6	619455	28997507	1621813676
7	34020	1592811	89086800
8	1410	66132	3699531
9	45	2145	120219
10	1	55	3135
11		1	66
12			1

Suggestion 2. Is it true that $a_{n1} = a_{n2}$ for $n \geq 2$? Can you write a special-purpose programme that will handle (2) for $n \geq 13$? What is the largest value of n that your special-purpose programme will reach?

The use of generating functions, such as

$$\sum_{n=0}^{\infty} p_n(y)\lambda^n \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{p_n(y)\lambda^n}{n!}$$

or, maybe, some other expression, is a well-worn technique for handling recurrence relations, especially when convolutions are present. Is there a convolution implicit in (2)?

Suggestion 3. Can you find a generating function suited to a solution of (2)?

If we write (2) in the form

$$(-1)^n \left(\frac{1}{4} - y\right)^n = \sum_{m=0}^n \binom{2n}{2m} \left(-\frac{1}{4}\right)^{n-m} p_m(y) \quad (3)$$

and multiply both sides by $\lambda^{2n}/(2n)!$ and sum from $n = 0$ to ∞ , we get

$$\cos(\lambda\sqrt{\frac{1}{4}-y}) = \left(\sum_{n=0}^{\infty} p_n(y) \frac{\lambda^{2n}}{(2n)!} \right) \cos \frac{1}{2}\lambda. \quad (4)$$

Hence a suitable generating function is

$$f(\lambda, y) = \sum_{n=0}^{\infty} p_n(y) \frac{\lambda^{2n}}{(2n)!} = \sec \frac{1}{2}\lambda \cos(\lambda\sqrt{\frac{1}{4}-y}). \quad (5)$$

Suggestion 4. The conjecture cannot be true in general unless it is true in the particular case $r = 1$, which will do for a modest start; so for brevity write $a_{n1} = b_n$. Can you find a generating function for b_n ? Can you find an asymptotic formula for b_n when $n \rightarrow \infty$? Can you prove that b_n is an odd positive integer? The square root in (5) is a bit of a pain in the neck: can you find a transformation from y into some other variable x that will make things simpler?

Finding a generating function for b_n is easy:

$$\sum_{n=1}^{\infty} b_n \frac{\lambda^{2n}}{(2n)!} = \left[\frac{\partial f}{\partial y} \right]_{y=0} = \lambda \tan \frac{1}{2}\lambda. \quad (6)$$

As an aside we may remark that the generating function for the Bernoulli numbers is $\lambda \cot \frac{1}{2}\lambda$; so that our b_n are, in an appropriate sense, reciprocal to the Bernoulli numbers. Since

$$\lambda \frac{d}{d\lambda} (\lambda \tan \frac{1}{2}\lambda) = \lambda \tan \frac{1}{2}\lambda + \frac{1}{2}\lambda^2 (1 + \tan^2 \frac{1}{2}\lambda), \quad (7)$$

we have

$$\sum_{n=1}^{\infty} (2n-1) b_n \frac{\lambda^{2n}}{(2n)!} = \frac{1}{2}\lambda^2 + \frac{1}{2} \left(\sum_{n=1}^{\infty} b_n \frac{\lambda^{2n}}{(2n)!} \right)^2; \quad (8)$$

and hence

$$(2n-1) b_n = \frac{1}{2} \sum_{m=1}^{n-1} \binom{2n}{2m} b_m b_{n-m} \quad (n \geq 2). \quad (9)$$

The numbers b_n can easily be calculated from the recursion (9), starting from $b_1 = 1$. We can write (9) as

$$(2n-1) b_n = \begin{cases} \sum_{m=1}^{\frac{1}{2}(n-1)} \binom{2n}{2m} b_m b_{n-m} & (n \text{ odd}), \\ \binom{2n-1}{n-1} b_{\frac{1}{2}n}^2 + \sum_{m=1}^{\frac{1}{2}(n-2)} \binom{2n}{2m} b_m b_{n-m} & (n \text{ even}). \end{cases} \quad (10)$$

Also (2) gives the alternative recursion formula

$$4^{n-1}b_n = (-1)^{n-1}n - \sum_{m=1}^{n-1} \binom{2n}{2m} (-1)^{n-m} 4^{m-1} b_m. \quad (11)$$

For the sake of induction on n , assume that b_1, b_2, \dots, b_{n-1} are integers. Then $(2n-1)b_n$ and $4^{n-1}b_n$ are also integers and, hence,

$$b_n = n^{2n-2}(4^{n-1}b_n) - (2n-1)b_n \sum_{m=0}^{2n-3} (2n)^m \quad (12)$$

is an integer. If b_1, b_2, \dots, b_{n-1} are odd integers, we have from (10) the congruence (modulo 2)

$$b_n \equiv \begin{cases} \sum_{m=1}^{\frac{1}{2}(n-1)} \binom{2n}{2m} & (n \text{ odd}) \\ \binom{2n-1}{n-1} + \sum_{m=1}^{\frac{1}{2}(n-2)} \binom{2n}{2m} & (n \text{ even}) \end{cases} \\ \equiv -1 + \frac{1}{2} \sum_{m=0}^n \binom{2n}{2m} = 2^{2n-2} - 1, \quad (13)$$

which shows that b_n is also odd. Finally (9) shows that b_n is positive when b_1, b_2, \dots, b_{n-1} are positive. This completes the inductive proof that b_n is an odd positive integer for all $n \geq 1$.

The right-hand side of (6) has a radius of convergence $|\lambda| = \pi$ and therefore

$$\limsup_{n \rightarrow \infty} \left(\frac{b_n}{(2n)!} \right)^{1/n} = \pi^{-2}. \quad (14)$$

However (14) is not accurate enough for an asymptotic formula for b_n as $n \rightarrow \infty$. Finding an exact formula for b_n , that will yield a good asymptotic formula, is an excellent test of whether an undergraduate can select the right technique from amongst those bits of information learnt from lectures. Once it is realized that the appropriate technique is contour integration, the procedure is straightforward, because $b_n/(2n)!$ is the residue of $\sin \frac{1}{2}z / (z^{2n} \cos \frac{1}{2}z)$ at $z = 0$. The integral of this function, taken round a square with vertices at $z = (\pm 1 \pm i)2m\pi$, tends to zero as $m \rightarrow \infty$ and, consequently, the sum of all its residues in the complex plane is zero. A simple calculation then gives, for all $n \geq 1$,

$$b_n = (2n)! \frac{4}{\pi^{2n}} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2n}} = (2n)! \frac{4S_n}{\pi^{2n}}, \quad (15)$$

where the sum S_n is defined by (15). Since S_n tends to 1 very rapidly as $n \rightarrow \infty$, (15) provides a very good asymptotic expression for b_n .

A convenient transformation, that removes the square root in (5), is

$$y = x - x^2. \quad (16)$$

This converts each $p_n(y)$ into a polynomial $q_n(x)$ of degree $2n$,

$$p_n(y) = q_n(x), \quad (17)$$

and (5) becomes

$$\begin{aligned} f(\lambda, y) = g(\lambda, x) &= \sum_{n=0}^{\infty} q_n(x) \frac{\lambda^{2n}}{(2n)!} \\ &= \sec \frac{1}{2}\lambda \cos \lambda \left(\frac{1}{2} - x\right) = \cos \lambda x + (\lambda^{-1} \sin \lambda x)(\lambda \tan \frac{1}{2}\lambda). \end{aligned} \quad (18)$$

Suggestion 5. Show that all the coefficients in $q_n(x)$ are integers and deduce that the same is true for $p_n(y)$.

Equating coefficients of $\lambda^{2n}/(2n)!$ in (18) and using (6), we have

$$\begin{aligned} q_n(x) &= (-x^2)^n + \sum_{m=0}^{n-1} \binom{2n}{2m} \frac{(-1)^m x^{2m+1}}{2m+1} b_{n-m} \\ &= (-x^2)^n + \frac{1}{2n+1} \sum_{m=0}^{n-1} \binom{2n+1}{2m+1} (-1)^m x^{2m+1} b_{n-m}. \end{aligned} \quad (19)$$

Since all the b_{n-m} in (19) are integers, we see that $(2n+1)q_n(x)$ is a polynomial with integer coefficients. On the other hand, substitution of (17) into (2) shows that $4^n q_n(x)$ is also a polynomial with integer coefficients; whereupon a straightforward modification of the argument in (12) proves that $q_n(x)$ has integer coefficients. Hence $p_n(y)$ has integer coefficients; for otherwise there would be a least value of r for each fixed n such that a_{nr} is not an integer, and the corresponding coefficient of x^r would not be an integer.

Suggestion 6. Can you find an equation connecting $p_{n-1}(y)$ with the first two derivatives of $p_n(y)$? Can you deduce that $a_{nr} > 0$ ($1 \leq r \leq n$)?

By (16) and (18)

$$(1-4y) \frac{\partial^2 f}{\partial y^2} - 2 \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 g}{\partial x^2} = -\lambda^2 g = -\lambda^2 f. \quad (20)$$

Equating coefficients of λ^{2n} in (20) yields the difference-differential equation

$$(4y-1)p_n''(y) + 2p_n'(y) = 2n(2n-1)p_{n-1}(y); \quad (21)$$

and hence, from the coefficients of y^{r-1} in (21), the partial difference equation

$$2r(2r-1)a_{nr} = r(r+1)a_{n,r+1} + 2n(2n-1)a_{n-1,r-1} \quad (1 \leq r < n). \quad (22)$$

Since $a_{nn} = 1$ for all $n \geq 1$, we can use (22) to calculate all the a_{nr} in the order $a_{21}, a_{32}, a_{31}, \dots, a_{n,n-1}, a_{n,n-2}, \dots, a_{n1}, \dots$; and in this process each a_{nr} is a positive function of previously calculated positive coefficients. Hence $a_{nr} > 0$ ($1 \leq r \leq n$) and we have now proved that all these coefficients are positive integers. The relation (22) provides an easy method for actually calculating a_{nr} ($1 \leq r \leq n \leq 20$ or 30 , say) on a computer: the only limitation is that the numbers a_{nr} sooner or later become unmanageably large. This calculation also contains useful intermediate checks $a_{n1} = b_n$ against values computed from

(9) or (11). The case $r = 1$ in (22) gives an affirmative answer to the query $a_{n1} = a_{n2}$ in Suggestion 2.

Suggestion 7. Generating functions afford a standard technique for solving linear partial difference equations. Can you solve (22) in this fashion?

Equation (19) gives an explicit formula for $q_n(x)$. Can we use it to obtain an explicit formula for $p_n(y)$ in terms of the coefficients b_n ($n = 1, 2, \dots$)?

Suggestion 8. More generally, given any polynomial $Q(x)$, is there a procedure for determining whether or not there exists a polynomial $P(y) = Q(x)$ connected with it under the transformation (16) and for calculating P when it does exist? Note that (16) is a (1,2)-correspondence, since it is linear in y but quadratic in x ; does this imply a possible ambiguity in the correspondence between P and Q ?

Consider first the numerical example

$$Q(x) = 4 + 3x - 10x^2 + 13x^3 + x^4 - 23x^5 + 31x^6 - 20x^7 + 5x^8, \quad (23)$$

and write down the coefficients of Q in the first row of the table below:

4	3	-10	13	1	-23	31	-20	5	0	0	...
	3	-7	6	7	-16	15	-5	0	0	0	...
		-7	-1	6	-10	5	0	0	0	0	...
			-1	5	-5	0	0	0	0	0	...
				5	0	0	0	0	0	0	...
					0	0	0	0	0	0	...

The first entry in the second row is a copy of the entry above it. Each remaining entry in the second row is the sum of the entry above it and the entry to the left of it: thus $-7 = -10 + 3$, $6 = 13 - 7$, etc. Subsequent rows are obtained from their preceding rows by the same rule, until we reach a row consisting entirely of zeros. The leading elements in the rows of this table give the polynomial

$$P(y) = 4 + 3y - 7y^2 - y^3 + 5y^4 \quad (24)$$

which corresponds to $Q(x)$ in (23) under (16). On the other hand, had $Q(x)$ been a polynomial for which no polynomial $P(y)$ existed, the procedure would not have terminated with a row of zeros.

The ballot numbers, so-called because of their appearance in the mathematical theory of elections, are the numbers in the infinite array

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1	1	1	1	1	1	1	1	...
	1	2	3	4	5	6	7	...
		2	5	9	14	20	27	...
			5	14	28	48	75	...
				14	42	90	165	...
					42	132	297	...
						132	429	...
							429	...

formed by the foregoing rules from a first row whose entries are all 1.

Suggestion 9. Can you explain and validate the calculation from (23) to (24)? Can you find a formula for the ballot number β_{ij} in the i th row and j th column of the array (25) for $1 \leq i \leq j$?

Given a polynomial $P(y)$, the transformation (16) uniquely determines a corresponding polynomial $Q(x) = P(x-x^2)$. On the other hand, the inverse of (16) is two-valued:

$$x = \frac{1}{2} \pm \frac{1}{2} \sqrt{1-4y}; \quad (26)$$

and thus, given a polynomial $Q(x)$, (26) determines a pair of functions

$$Q\left(\frac{1}{2} \pm \frac{1}{2} \sqrt{1-4y}\right) = P(y) \pm R(y) \sqrt{1-4y}, \quad (27)$$

where P and R are polynomials in y . But the right-hand side of (27) is a polynomial if and only if $R = 0$. Hence, if there exists a polynomial $P(y) = Q(x)$, the choice of alternative sign in (26) is immaterial. Thus in all cases we may take the inverse of (16) to be

$$x = \frac{1}{2} - \frac{1}{2} \sqrt{1-4y} = y + y^2 + 2y^3 + 5y^4 + 14y^5 + \dots \quad (28)$$

on expanding the square root by the binomial theorem for $|y| < \frac{1}{4}$.

The rules for generating the array (25) can be summarized as

$$\beta_{1j} = 1 \quad (j \geq 1); \quad (29)$$

$$\beta_{ii} = \beta_{i-1,i} \quad (i \geq 2); \quad (30)$$

$$\beta_{ij} = \beta_{i-1,j} + \beta_{i,j-1} \quad (j > i > 1). \quad (31)$$

We shall now prove that, for $j \geq 1$,

$$x^j = \sum_{i=0}^{\infty} \beta_{i+1,i+j} y^{i+j}. \quad (32)$$

This is the j th power of the series (28); so it commences with the term y^j ; and, accordingly, with $i = 0$ in (32), we have $\beta_{1j} = 1$, which satisfies the rule (29). Next, from (16),

$$\sum_{i=0}^{\infty} \beta_{i+1,i+2} y^{i+2} = x^2 = x - y = \sum_{i=1}^{\infty} \beta_{i+1,i+1} y^{i+1} \quad (33)$$

and comparison of the coefficients of y^{i+2} in (33) confirms the rule (30). Similarly, we confirm the rule (31) by comparing coefficients in the expansion of the identity

$$x^j = x^{j+1} + yx^{j-1}. \quad (34)$$

Having established that (32) conforms with the rules (29), (30) and (31), we can assert that the expansion (28) does indeed represent the elements on main diagonal of (25): that is to say

$$\beta_{ii} = \frac{1}{i} \binom{2i-2}{i-1}, \quad (35)$$

since this is the general coefficient in the expansion (28) and not merely one of

the few in (25) as far as it was written out. Returning to (25) and its rules of formation, we see that the elements 1, 3, 9, ... in the second superdiagonal are the first differences of the elements 1, 2, 5, ... in the first superdiagonal; and likewise the elements in the k th superdiagonal are the first differences of the elements in the $(k-1)$ th superdiagonal ($k \geq 2$). By writing down the first, second, third, ... differences of the sequence (35) and noting how they behave, the formula

$$\beta_{ij} = \frac{j+1-i}{j} \binom{i+j-2}{j-1} \quad (1 \leq i \leq j) \quad (36)$$

is readily suggested; and the truth of (36) is checked by confirming that it satisfies the rules (29), (30) and (31). Of course, anyone with enough intuition to spot (36) straightaway would not need to go through the investigatory arguments from (32) to (35) as one of the possible ways of arriving at (36). As a matter of fact, they were the means by which I myself reached (36): and it was not until I had got to (36) via (32) to (35) that I recognized that I was dealing with the ballot numbers, this recognition being stimulated by writing this part of the paper on 11 June 1987, the day of the British general election.

It remains to explain why the procedure from (23) to (24) works. When I was a schoolboy doing elementary algebra and dividing one polynomial by another, we all had to learn how to find quotients and remainders by the method of detached coefficients (which should be self-explanatory terminology). The procedure given is simply this method applied to repeated division of $Q(x)$ by $x-x^2$. The array (25), if preceded by an initial row 0 1 0 0 0 ... is the corresponding procedure for $Q(x) = x$ and (32) is the procedure for $Q(x) = x^j$, which, by the linearity of vector spaces, must yield numbers embedded somewhere in the array (25), and it is not difficult to see that the choice of suffices and indices in (32) gives the appropriate embedding.

Finally by (32), if (16) transforms a polynomial $P(y)$ into

$$Q(x) = \sum_j Q_j x^j,$$

then

$$\begin{aligned} P(y) &= \sum_r P_r y^r = Q_0 + \sum_{j>0} \sum_{i=0}^j Q_j \beta_{i+1, i+j} y^{i+j} \\ &= Q_0 + \sum_{j>0} \sum_{i=0}^j \frac{j Q_j}{i+j} \binom{2i+j-1}{i+j-1} y^{i+j}. \end{aligned} \quad (37)$$

Hence

$$P_r = \sum_{j=1}^r \frac{j Q_j}{r} \binom{2r-j-1}{r-1} \quad (r \geq 1); \quad (38)$$

and (19) and (38) give

$$a_{nr} = \frac{1}{r} \sum_{m=0}^{[\frac{1}{2}(r-1)]} \binom{2n}{2m} \binom{2r-2m-2}{r-1} (-1)^m b_{n-m} \quad (1 \leq r \leq n) \quad (39)$$

as the required solution of the partial difference equation (22). In (39), $[\frac{1}{2}(r-1)]$ denotes, as usual, the integer part of $\frac{1}{2}(r-1)$.

Of course, this is not the only way of solving (22); and for an alternative method we return to Suggestion 7. Early on in the paper, you were warned that some of the suggestions might be misleading and in one sense this is true of Suggestion 7. It is certainly possible to tackle partial difference equations with generating functions, but, in the present case, this merely leads back to (5). So we need an alternative manipulation of (5) and this is provided by the Euler numbers e_n defined by the expansion

$$\sec \lambda = \sum_{n=0}^{\infty} e_n \frac{\lambda^{2n}}{(2n)!} \quad (|\lambda| < \frac{1}{2}\pi). \quad (40)$$

Suggestion 10. Can you prove that the Euler numbers are odd positive integers and can you find an exact formula for e_n and an asymptotic formula as $n \rightarrow \infty$? Can you find a formula for a_{nr} in terms of the Euler numbers?

The identity

$$1 = \cos \lambda \sec \lambda = \left(\sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n)!} \right) \left(\sum_{n=0}^{\infty} e_n \frac{\lambda^{2n}}{(2n)!} \right) \quad (41)$$

yields the recurrence relation

$$e_n = \sum_{m=0}^{n-1} (-1)^{n-1-m} \binom{2n}{2m} e_m \quad (n \geq 1) \quad (42)$$

starting from $e_0 = 1$. Successive values of e_n can be calculated from (42), which (by induction on n) shows that the Euler numbers are odd integers because

$$\sum_{m=0}^{n-1} \binom{2n}{2m} = 2^{2n-1} - 1 \equiv 1 \pmod{2} \quad (n \geq 1). \quad (43)$$

The first few Euler numbers are $e_0, e_1, \dots = 1, 1, 5, 61, 1385, 50521, \dots$. Contour integration, as in (15), provides an exact formula

$$e_n = (2n)! \frac{4^{n+1}}{\pi^{2n+1}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{2n+1}} = (2n)! \frac{4^{n+1} T_n}{\pi^{2n+1}} \quad (n \geq 0), \quad (44)$$

where the sum T_n , defined by (44), satisfies $T_n > 1 - 3^{-(2n+1)} > 0$, thus proving that $e_n > 0$ for all n .

We can write (5) as

$$\sum_{n=0}^{\infty} p_n(y) \frac{\lambda^{2n}}{(2n)!} = \left(\sum_{n=0}^{\infty} \frac{e_n}{4^n} \frac{\lambda^{2n}}{(2n)!} \right) \left(\sum_{n=0}^{\infty} (y - \frac{1}{4})^n \frac{\lambda^{2n}}{(2n)!} \right), \quad (45)$$

which gives

$$p_n(y) = \frac{1}{4^n} \sum_{m=0}^n \binom{2n}{2m} (4y-1)^{n-m} e_m \quad (n \geq 0) \quad (46)$$

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and thence

$$a_{nr} = (-\frac{1}{4})^{n-r} \sum_{m=0}^{n-r} \binom{2n}{2m} \binom{n-m}{r} (-1)^m e_m \quad (1 \leq r \leq n). \quad (47)$$

The two alternative solutions (39) and (47) of the partial difference equation (22) are of course equal to each other and this provides a set of identities relating the numbers b_{n-m} and the numbers e_m . The solution (39) is useful for calculating a_{nr} when r is small, whereas (47) is useful when $n-r$ is small. In fact, writing $r = n-k$ in (47), we get

$$a_{n,n-k} = (-\frac{1}{4})^k \sum_{m=0}^k \binom{2n}{2m} \binom{n-m}{k-m} (-1)^m e_m \quad (0 \leq k \leq n). \quad (48)$$

In particular

$$a_{n,n-1} = \binom{n}{2} \quad (n \geq 1); \quad (49)$$

$$a_{n,n-2} = 3\binom{n}{3} + 5\binom{n}{4} \quad (n \geq 2); \quad (50)$$

$$a_{n,n-3} = 17\binom{n}{4} + 70\binom{n}{5} + 61\binom{n}{6} \quad (n \geq 3). \quad (51)$$

These results suggest that there might be a formula of the type

$$a_{n,n-k} = \sum_{j=1}^k c_{jk} \binom{n}{j+k} \quad (n \geq k). \quad (52)$$

Suggestion 11. Can you prove (52)? Can you obtain a partial difference equation for c_{jk} by substituting (52) into (22)? Are the numbers c_{jk} ($1 \leq j \leq k$) all positive integers? Can you identify c_{1k} and c_{kk} ?

This paper is an exercise in manipulation and the substitution of (52) into (22) offers a worthwhile opportunity for practising manipulation of algebraic formulae. Manipulations can be classified by borrowing the terminology of U.S. boxing weights:

Heavyweight	no limit
Cruiserweight	81 kg
Middleweight	75 kg
Light middleweight	71 kg
Welterweight	67 kg
Light welterweight	63½ kg
Lightweight	60 kg
Featherweight	57 kg
Bantamweight	54 kg
Flyweight	51 kg
Light flyweight	48 kg

Undergraduate courses rarely go beyond the flyweight class in manipulation; but

the professional mathematician often needs to wrestle with quite heavy algebraic expressions. For heavyweight manipulations special algebraic techniques are necessary: a typical example is the technique of algebraic pattern functions discussed in Chapter 12 of Kendall and Stuart (1969). The present case of determining a partial difference equation by substituting (52) into (22) is much less demanding, say light welterweight, but nevertheless it calls for a modest degree of manipulative skill, which will be described presently. First however you should see how much you can manage on your own before reading the hints given below.

Equation (48) is stated under the condition ($0 \leq k \leq n$). However, for a fixed $k \geq 1$, we can regard the right-hand side of (48) as a function defined for all n . In this light $\binom{2n}{2m}$ is a polynomial of degree $2m$ in n , and $\binom{n-m}{k-m}$ is a polynomial of degree $k-m$ in n . So the right-hand side of (48) is a polynomial of degree $2k$ in n . Also, whatever the value of m subject to $0 \leq m \leq k$, the product $\binom{2n}{2m}\binom{n-m}{k-m}$ is divisible by the factor $n(n-1)\dots(n-k+1)$. When $n = k \geq 1$, the right-hand side of (48) is

$$\left(-\frac{1}{4}\right)^k \sum_{m=0}^k \binom{2n}{2m} (-1)^m e_m = 0 \quad (53)$$

by (42). Hence $a_{n,n-k}$ is a polynomial of degree $2k$ in n , which vanishes whenever $n = 0, 1, \dots, k$. Consequently $a_{n,n-k}$ can be written in the form (52) for some constants c_{jk} . Now put $n = k+1$ in (52). We get

$$c_{1k} = a_{k+1,1} = b_{k+1}; \quad (54)$$

so c_{1k} is a positive integer. Next put $n = h+k$ ($1 < h \leq k$) in (52):

$$c_{hk} = a_{h+k,k} - \sum_{j=1}^{h-1} c_{jk} \binom{h+k}{j+k}. \quad (55)$$

If $c_{1k}, c_{2k}, \dots, c_{h-1,k}$ are integers, so is c_{hk} . Thus, by induction on h , all the c_{jk} are integers. Comparing coefficients of n^{2k} in (48) and (52), we find that

$$c_{kk} = e_k. \quad (56)$$

I shall now describe how the substitution of (52) into (22) leads to the partial difference equation

$$j(2j-1)c_{jk} = (j+k)[2(k-j+1)c_{j-1,k} + \frac{1}{2}(j+k-1)c_{j-1,k-1} + jc_{j,k-1}] \\ + \frac{1}{2}j(j+1)c_{j+1,k-1} \quad (1 \leq j \leq k) \quad (57)$$

with the convention that $c_{0k} = c_{k,k-1} = c_{k+1,k-1} = 0$. The actual calculation should easily fit on half a sheet of paper, though it will take more than that to describe it here. The essential thing in manipulation is to use an abbreviated notation and never to write down more than is barely necessary, because too many symbols cluttering up the working will inevitably cause mistakes. The basic idea behind the calculation is to compare coefficients of the binomial quantities $\binom{n}{j+k}$ in (52). However $j+k$ is too elaborate to keep repeating; so we replace the right-hand side of (52) by

$$c_{\alpha-k,k} \binom{n}{\alpha}, \quad (58)$$

the symbol α operating as a dummy over which summation is tacitly understood. Dummy summation further cleans up and lightens the notation since we no longer have to carry a \sum sign in (58) and subsequent equations. We also arrange the calculation so that the second suffix k in (58) shall take the value k on the left-hand side of the manipulation and the value $k-1$ on the right-hand side. With this understanding, there will be no need to write down the second suffix; so (58) appears in the simpler form

$$c_{\alpha-k} \binom{n}{\alpha}. \quad (59)$$

To start the calculation, we take $r = n-k$ and write (22) in the form

$$\begin{aligned} (2n-1)n(a_{n,n-k} - a_{n-1,n-1-k}) + k(2k+1-4n)a_{n,n-k} \\ = \frac{1}{2}[n^2 - (2k-1)n + k(k-1)]a_{n,n-(k-1)}. \end{aligned} \quad (60)$$

Notice how, as previously mentioned, the second suffix uses k on the left of (60) and $k-1$ on the right of (60). The expression $n(a_{n,n-k} - a_{n-1,n-1-k})$ has been isolated to take advantage of the identity

$$n \left[\binom{n}{\alpha} - \binom{n-1}{\alpha} \right] = \alpha \binom{n}{\alpha}. \quad (61)$$

We also need

$$n \binom{n}{\alpha} = (\alpha+1) \binom{n}{\alpha+1} + \alpha \binom{n}{\alpha} \quad (62)$$

and its iterate

$$n^2 \binom{n}{\alpha} = (\alpha+1)(\alpha+2) \binom{n}{\alpha+2} + (\alpha+1)(2\alpha+1) \binom{n}{\alpha+1} + \alpha^2 \binom{n}{\alpha}. \quad (63)$$

Using (61) and (62), we can now write down the left-hand side of (60) as

$$\begin{aligned} c_{\alpha-k} \left[\left\{ 2(\alpha+1) \binom{n}{\alpha+1} + 2\alpha \binom{n}{\alpha} - \binom{n}{\alpha} \right\} \alpha + k(2k+1) \binom{n}{\alpha} - 4k(\alpha+1) \binom{n}{\alpha+1} - 4k\alpha \binom{n}{\alpha} \right] \\ = c_{\alpha-k} \left[2(\alpha+1)(\alpha-2k) \binom{n}{\alpha+1} + (\alpha-k)(2\alpha-2k-1) \binom{n}{\alpha} \right] \\ = [2\alpha(\alpha-2k-1)c_{\alpha-k-1} + (\alpha-k)(2\alpha-2k-1)c_{\alpha-k}] \binom{n}{\alpha} \end{aligned} \quad (64)$$

because $\alpha+1$ can be shifted to α , being a dummy. Putting $\alpha = j+k$ in (64) and restoring the second suffix, the coefficient of $\binom{n}{j+k}$ on the left-hand side of (60) is the left-hand side of

$$\begin{aligned} j(2j-1)c_{jk} - 2(j+k)(k-j+1)c_{j-1,k} \\ = (j+k) \left[\frac{1}{2}(j+k-1)c_{j-1,k-1} + jc_{j,k-1} \right] + \frac{1}{2}j(j+1)c_{j+1,k-1} \end{aligned} \quad (65)$$

whose right-hand side equals the corresponding expression obtained from the

right-hand side of (60) using (62) and (63); and this gives (57). I have not given here the detailed working for the right-hand side of (65), because you may find it useful manipulative practice to provide it yourself by writing down the equations that correspond to (64).

We have already proved that all the coefficients c_{jk} are integers, and we can now show that they are positive. Starting from $c_{11} = 1$, given by (49), we can successively calculate $c_{12}, c_{22}, \dots, c_{1k}, c_{2k}, \dots, c_{kk}, \dots$ in that order from (57). At each stage of this calculation, the right-hand side of (57) consists of non-negative quantities previously determined at an earlier stage, and at least one of the terms on the right-hand side of (57) will be strictly positive. Hence $c_{jk} > 0$ by double induction on j and k . The calculation contains useful periodic checks (54) and (56) against the values of b_{k+1} and e_k already found from (9), (11) and (42); and it can be programmed on a computer to yield the typical results shown in Table 2.

Table 2. Specimen values of c_{jk} .

j	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
1	1	3	17	155	2 073	38 227	929 569
2		5	70	1 143	23 716	623 753	20 454 498
3			61	2 317	82 286	3 243 270	147 624 355
4				1 385	110 556	7 258 090	484 343 420
5					50 521	7 293 671	796 220 487
6						2 702 765	639 139 202
7							199 360 981
j	$k = 8$		$k = 9$		$k = 10$		
1	28 820 619		1 109 652 905		51 943 281 731		
2	821 446 715		39 737 099 776		2 281 831 661 709		
3	7 822 241 867		481 437 876 668		34 219 261 121 916		
4	34 986 532 875		2 798 726 703 448		249 890 883 433 044		
5	83 060 436 121		8 943 001 405 982		1 026 519 528 762 378		
6	107 687 984 425		16 561 891 136 560		2 534 516 719 669 830		
7	71 970 566 865		17 708 437 066 700		3 838 877 672 220 204		
8	19 391 512 145		10 138 324 694 008		3 490 455 615 793 668		
9			2 404 879 675 441		1 747 839 965 669 899		
10					370 371 188 237 525		

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Equations (39) and (47) are not well suited as starting points for an asymptotic formula for a_{nr} as $n \rightarrow \infty$, because they both contain the oscillatory factor $(-1)^m$. On the other hand, (52) consists only of positive terms and so offers good prospects, provided that we can get results about the sizes of the coefficients c_{jk} .

Suggestion 12. Can you find good inequalities for the coefficients c_{jk} ($1 \leq j \leq k$)?

The partial difference equation (57) is linear and, accordingly, you might like to try solving it by means of a generating function. However this is a didactic paper whose purpose is to illustrate a variety of manipulative techniques and so it is appropriate instead to describe a different approach, called the *method of monotone normalization*, which is applicable to difference equations that are

known to have positive solutions. Suppose that we seek a solution $\gamma_{jk} > 0$ of a (possibly non-linear) partial difference equation which can be written in the form

$$\phi_{jk}\gamma_{jk} - \psi_{jk}\gamma_{j-1,k} = F_{jk}(\gamma_{st}, \gamma_{uv}, \dots) > 0, \quad (66)$$

where ϕ_{jk} and ψ_{jk} are known positive functions of j and k and F_{jk} is some positive monotone function of its arguments $\gamma_{st}, \gamma_{uv}, \dots$. Together with a function ω_k to be chosen presently, we then define a function χ_{jk} , called the *monotone normalizer*, by

$$\chi_{jk} = \omega_k \prod_{i=1}^j \frac{\psi_{ik}}{\phi_{ik}} > 0 \quad (67)$$

and we make the transformation

$$\gamma_{jk} = \chi_{jk}\delta_{jk} \quad (68)$$

in (66) to get

$$\delta_{jk} - \delta_{j-1,k} = (\phi_{jk}\chi_{jk})^{-1}F_{jk} > 0. \quad (69)$$

Thus δ_{jk} is an increasing function of j for each fixed k and, accordingly, $\delta_{1k} \leq \delta_{jk} \leq \delta_{kk}$ ($1 \leq j \leq k$). Suppose further that the quantities δ_{1k} and δ_{kk} can be calculated from the boundary conditions associated with (66) and that a suitable choice of the function ω_k in (67) makes the difference $\delta_{kk} - \delta_{1k}$ reasonably small in comparison with the quantities δ_{jk} . Then we have a satisfactory inequality

$$\chi_{jk}\delta_{1k} \leq \gamma_{jk} \leq \chi_{jk}\delta_{kk}, \quad (1 \leq j \leq k). \quad (70)$$

In favourable circumstances this inequality can now be fed back into the right-hand side of (69), leading to an improved version of (70), and iteration of this process can yield an exact solution of (66) or else bounds of arbitrarily tight accuracy. We now apply this technique of monotone normalization to (57).

In (57) make the substitution

$$c_{jk} = \binom{j+k}{2j} 4^{j-k} e_k d_{jk}, \quad (71)$$

to get

$$d_{jk} - d_{j-1,k} = \frac{4e_{k-1}}{e_k} \left(\frac{1}{4}d_{j-1,k-1} + \frac{k-j}{2j-1}d_{j,k-1} + \frac{(k-j)(k-j-1)}{(2j-1)(2j+1)}d_{j+1,k-1} \right). \quad (72)$$

Thus $d_{1k}, d_{2k}, \dots, d_{kk}$ is an increasing sequence because the right-hand side of (72) is positive. By (15), (44), (54), (56) and (71)

$$\frac{2}{\pi} \leq \frac{2}{\pi} \left(1 + \frac{1}{2k} \right) \frac{S_{k+1}}{T_k} = \frac{2^{2k-1}b_{k+1}}{k(k+1)e_k} = d_{1k} \leq d_{jk} \leq d_{kk} = 1. \quad (73)$$

We can now feed (73) back into (72):

$$\begin{aligned} d_{jk} - d_{j-1,k} &\leq \frac{4e_{k-1}}{e_k} \left(\frac{1}{4} + \frac{k-j}{2j-1} + \frac{(k-j)(k-j-1)}{(2j-1)(2j+1)} \right) \\ &= \frac{e_{k-1}(4k^2-1)}{e_k(4j^2-1)} = \frac{\pi^2 T_{k-1}}{8T_k} \left(\frac{2k+1}{2k} \right) \left(\frac{1}{2j-1} - \frac{1}{2j+1} \right). \end{aligned} \quad (74)$$

Now sum (74) over j from $j = i+1$ to $j = k$ to get

$$1 - \frac{\pi^2(k-i)}{8k(2i+1)} \leq d_{ik} \quad (1 \leq i \leq k) \quad (75)$$

because $T_{k-1} < T_k$. For $k > 1$, the lower bound in (73) leads to an upper bound for d_{ik} . A little care is appropriate here because $c_{k,k-1} = c_{k+1,k-1} = 0$, by definition, and therefore we ought to have $d_{k,k-1} = d_{k+1,k-1} = 0$ when they occur in (72). However, when they do occur in (72), they are multiplied by zero factors; so it will make no difference if we use (73) in the form

$$\frac{2(2k-1)S_k}{\pi(2k-2)T_{k-1}} \leq d_{j,k-1} \quad (1 < j \leq k+1 > 2) \quad (76)$$

when feeding it back throughout (72). We then obtain, after summation from $j = 2$ to $j = i$,

$$d_{ik} \leq 1 - \frac{\pi S_k(2k-1)(k-i)}{4T_k(2k-2)k(2i+1)} \leq 1 - \frac{\pi(k-i)}{4k(2i+1)} \quad (1 \leq i \leq k > 1). \quad (77)$$

We may remove the restriction $k > 1$ in (77), because (77) is trivially true when $i = k = 1$. It is possible to sharpen the inequalities (75) and (77) by feeding them back into (72); but we shall not do that here since they are sharp enough in their present form for what comes next. Writing j for i in (75) and (77), we can conclude from (71) that

$$\left(1 - \frac{\pi^2(k-j)}{8k(2j+1)} \right) \binom{j+k}{2j} 4^{j-k} e_k \leq c_{jk} \leq \left(1 - \frac{\pi(k-j)}{4k(2j+1)} \right) \binom{j+k}{2j} 4^{j-k} e_k \quad (78)$$

holds for all $1 \leq j \leq k$. From (52) and (78) we have

$$\frac{e_k}{4^k} \left(U - \frac{\pi^2 V}{8k} \right) \leq a_{n,n-k} \leq \frac{e_k}{4^k} \left(U - \frac{\pi V}{4k} \right), \quad (79)$$

where

$$U = \sum_{j=1}^k \binom{n}{j+k} \binom{j+k}{2j} 4^j, \quad V = \sum_{j=1}^{k-1} \binom{n}{j+k} \binom{j+k}{2j+1} 4^j. \quad (80)$$

Suggestion 13. Can you sum the series U and V in (80)?

In the identity

$$(1+z)^{2n} = \sum_{s=0}^n \binom{n}{s} (1+2z)^s z^{2(n-s)} = \sum_{s=0}^n \sum_{t=0}^s \binom{n}{s} \binom{s}{t} 2^t z^{2n-2s+t} \quad (81)$$

pick out the coefficient of z^{2n-2k} . Then t must be even, say $t = 2j$, and $s = j+k$. This gives

$$\binom{2n}{2k} = \sum_{j=0}^k \binom{n}{j+k} \binom{j+k}{2j} 4^j \quad (82)$$

and, hence,

$$U = \binom{2n}{2k} - \binom{n}{k}. \quad (83)$$

Similarly picking out the coefficient of $z^{2n-2k+1}$ and putting $t = 2j+1$, we find

$$\frac{V}{k} = \frac{1}{2n-2k+1} \binom{2n}{2k} - \binom{n}{k}. \quad (84)$$

With $r = n-k$, we get from (79), (83) and (84)

$$a_{nr} = \frac{e_{n-r}}{4^{n-r}} \left\{ \binom{2n}{2r} \left(1 - \frac{g_{nr}}{2r+1} \right) - \binom{n}{r} (1 - g_{nr}) \right\} \quad (1 \leq r < n), \quad (85)$$

for some quantities g_{nr} that satisfy

$$\frac{3}{4} < \frac{1}{4}\pi \leq g_{nr} \leq \frac{1}{8}\pi^2 < \frac{4}{3} \quad (1 \leq r < n). \quad (86)$$

Since

$$0 < \binom{n}{r} / \binom{2n}{2r} \leq \frac{1}{2n-1} \quad (1 \leq r < n), \quad (87)$$

we have

$$a_{nr} \approx \frac{e_{n-r}}{4^{n-r}} \binom{2n}{2r} \quad \text{as } n \rightarrow \infty \text{ and } r \rightarrow \infty; \quad (88)$$

and the error in this asymptotic formula is bounded by estimates provided by (85) and (86). However, (88) does not provide for the case when $n \rightarrow \infty$ while r remains finite; and to deal with this remaining case we define θ_{nr} by

$$a_{nr} = \frac{e_{n-r}}{4^{n-r}} \binom{2n}{2r} \theta_{nr} \quad (0 \leq r \leq n); \quad (89)$$

and we expect the existence of the limits

$$\theta_r = \lim_{n \rightarrow \infty} \theta_{nr} \quad (r \geq 0). \quad (90)$$

Suggestion 14. Can you prove that the limits (90) exist? Can you calculate θ_r ?

Substitute (89) into (22), and use (15) and (44) to get

$$\theta_{nr} - \theta_{n-1, r-1} = \frac{\pi^2 T_{n-r-1}}{4(4r^2-1) T_{n-r}} \theta_{n, r+1} \quad (1 \leq r < n) \quad (91)$$

and

$$\theta_{n0} = 0, \quad \theta_{n1} = \frac{2S_n}{\pi T_{n-1}}, \quad \theta_{nn} = 1. \quad (92)$$

Since $S_n \rightarrow 1$ and $T_n \rightarrow 1$ as $n \rightarrow \infty$, the limits in (90) exist for $r = 0$ and $r = 1$:

$$\theta_0 = 0, \quad \theta_1 = \frac{2}{\pi}. \quad (93)$$

Fix $r \geq 1$ and let $n \rightarrow \infty$ in (91). Then θ_{r+1} exists if θ_r and θ_{r-1} both exist. The existence of θ_r for all $r \geq 0$ now follows from (91) and (93) by induction on r and

$$\theta_{r+1} = \frac{4}{\pi^2}(4r^2 - 1)(\theta_r - \theta_{r-1}) \quad (r \geq 1). \quad (94)$$

From (85), (86) and (87) with fixed $r \geq 1$, we have

$$\begin{aligned} 1 - \frac{\pi^2}{8(2r+1)} &= \liminf_{n \rightarrow \infty} \left(1 - \frac{g_{nr}}{2r+1} \right) \leq \theta_r \\ &\leq \limsup_{n \rightarrow \infty} \left(1 - \frac{g_{nr}}{2r+1} \right) \leq 1 - \frac{\pi}{4(2r+1)}. \end{aligned} \quad (95)$$

Hence $\theta_{r+1} > 0$ and (94) gives $\theta_r > \theta_{r-1}$ ($r \geq 1$). Consequently from (93) and (95)

$$\frac{2}{\pi} = \theta_1 < \theta_2 < \dots < \theta_r \rightarrow 1 \quad \text{as } r \rightarrow \infty. \quad (96)$$

In principle, successive θ_r can be calculated from (94), starting from (93); and θ_r is a polynomial of degree $2r-1$ in $(2/\pi)$. For example

$$\theta_4 = 1575(2/\pi)^7 - 630(2/\pi)^5. \quad (97)$$

However, as r increases, the coefficients in these polynomials increase even more rapidly than the powers of $(2/\pi)$ decrease; so, for large r , θ_r is the sum of large quantities of opposite signs and the value of θ_r depends critically upon a highly accurate value of $2/\pi$. Thus, if we use $\pi = 3.14159$ rounded down to five places of decimals, we find that (94) yields $\theta_{20} = 8.08 \times 10^{31}$, whereas $\pi = 3.14160$ rounded up gives $\theta_{20} = -2.24 \times 10^{32}$. Even with π respectively rounded down and up to thirty places of decimals, we can only assert that $-1.51 \times 10^7 \leq \theta_{20} \leq 1.53 \times 10^7$, whereas we already know that $0.970 \leq \theta_{20} \leq 0.981$ from (95). Further details appear in Table 3 below.

Entries in the second column of Table 3 are the correct values of θ_r calculated from formula (103) correct to ten places of decimals. The incorrect values of θ_r appear as entries in the last six columns of this table and are of the typical form $808:32 = 0.808 \times 10^{32}$, $-224:33 = -0.224 \times 10^{33}$, etc. All such entries were calculated with quadruple-length (128 bits) floating-point arithmetic from the formula (94) but are only quoted to three significant places of decimals; and blank entries in the last six columns represent results that are correct to three significant decimals. Calculations in the column headed π_5^- were performed with $\pi = 3.14159$ rounded down to five places of decimals and, likewise, with $\pi = 3.14160$ rounded up to five places in the column headed π_5^+ . Similarly in the last four columns calculations used π rounded up or down to fifteen or thirty places of decimals.

† Table 3. Correct and incorrect values of θ_r .

r	θ_r	π_5^-	π_5^+	π_{15}^-	π_{15}^+	π_{30}^-	π_{30}^+
1	0.636 619 7724						
2	0.774 036 8264						
3	0.835 395 5140						
4	0.870 370 8798						
5	0.893 023 8569	896: 0	886: 0				
6	0.908 909 6752	101: 1	620: 0				
7	0.920 673 9896	681: 1	-154: 2				
8	0.929 739 9179	458: 3	-127: 4				
9	0.936 941 9967	467: 5	-129: 6				
10	0.942 802 3179	605: 7	-167: 8		941: 0		
11	0.947 664 3935	970: 9	-269: 10	103: 1	669: 0		
12	0.951 763 5686	189: 12	-523: 12	179: 2	-532: 2		
13	0.955 266 5275	438: 14	-121: 15	393: 4	-126: 5		
14	0.958 294 6508	119: 17	-330: 17	107: 7	-342: 7		
15	0.960 938 4388	377: 19	-104: 20	339: 9	-108: 10		
16	0.963 266 7512	137: 22	-379: 22	123: 12	-393: 12		
17	0.965 332 9141	566: 25	-157: 25	509: 14	-162: 15	107: 1	859: 0
18	0.967 178 8694	264: 27	-732: 27	238: 17	-759: 17	510: 2	-486: 2
19	0.968 838 0598	138: 30	-383: 30	124: 20	-397: 20	262: 5	-259: 5
20	0.970 337 4792	808: 32	-224: 33	726: 22	-232: 23	153: 8	-151: 8

Since incorrect values of θ_r originate from multiplying small differences $\theta_r - \theta_{r-1}$ by increasingly large factors $(4r^2 - 1)(2/\pi)^2$ in (94), it might seem sensible to calculate the θ_r by some backward-shooting technique. For example, suppose we take a pair of trial values θ_{n-1} and θ_n for some largish value of n , say $n = 20$, we could successively calculate $\theta_{19}, \theta_{18}, \dots$ from (94) until we reach θ_1 and θ_0 . Can we then iteratively adjust the trial starting values θ_{19} and θ_{20} to ensure that $\theta_1 = 2/\pi$ and $\theta_0 = 0$? A little experimentation on a computer should convince you that this idea does not work: in fact, although θ_1 can easily be made to equal $2/\pi$ by multiplying all θ_r by a suitable constant, the calculated value of θ_0 will remain obstinately indistinguishable from zero whatever the trial ratio θ_{19}/θ_{20} might have been. There are several other types of iterative possibility, but they are all liable to fail because there is a concealed eigenvalue problem associated with (94). In short, if we consider the more general recurrence relation, involving a constant $\xi \neq 0$,

$$\theta_{r+1} = \xi^{-2}(4r^2 - 1)(\theta_r - \theta_{r-1}), \quad \theta_0 = 0, \quad \theta_1 = \xi^{-1}, \quad (98)$$

instead of (93) and (94), it will emerge that the θ_r can only be bounded as $r \rightarrow \infty$ when ξ is an eigenvalue, of which $\xi = \frac{1}{2}\pi$ happens to be a solution.

Suggestion 15. Can you solve the eigenvalue problem enunciated above?

Since (98) is a second-order linear difference equation, its general solution is a linear combination of two independent solutions Θ_r and Φ_r ,

$$\theta_r = A\Theta_r + B\Phi_r, \quad (99)$$

where the constants A and B are to be chosen to satisfy the initial values $\theta_0 = 0$ and $\theta_1 = \xi^{-1}$. Instead of expressing Θ_r and Φ_r as polynomials in ξ^{-1} , we

express them as power series in ξ whose coefficients decrease much more rapidly than the powers of ξ increase. Thus, substituting $\Theta_r = \sum_n C_{nr} \xi^n$ in (98), we find that

$$\Theta_r = \sum_{s=0}^{\infty} \prod_{t=1}^s \frac{\xi^2}{2t(1-2t-2r)}, \quad (100)$$

upon equating coefficients of ξ^n , and similarly

$$\Phi_r = \left(\prod_{s=0}^{r-1} \frac{4s^2-1}{\xi^2} \right) \left(\sum_{s=0}^{\infty} \prod_{t=1}^s \frac{\xi^2}{2t(2r-2t-1)} \right). \quad (101)$$

In (100) and (101) an empty product, such as $\prod_{t=1}^0$ or $\prod_{s=0}^{-1}$ is interpreted as 1. In particular (100) and (101) give

$$\Theta_0 = \cos \xi, \quad \Theta_1 = \Phi_0 = \xi^{-1} \sin \xi, \quad \Phi_1 = -\xi^{-2} \cos \xi. \quad (102)$$

The first product in (101) tends rapidly to $-\infty$ as $r \rightarrow \infty$. So there will be a bounded solution of (98) if and only if $B = 0$ in (99). The initial conditions for θ_0 and θ_1 in (98) give $A = \sin \xi$ and $B = -\xi \cos \xi$. Hence $\cos \xi = 0$; and the appropriate eigenvalues of ξ are $\xi = \frac{1}{2}\pi + m\pi$, where m is an integer. The appropriate solution of (94) is therefore

$$\theta_r = \sum_{s=0}^{\infty} \prod_{t=1}^s \frac{\pi^2}{8t(1-2t-2r)}. \quad (103)$$

As already mentioned, (103) provides the correct values of θ_r in Table 3. As an aside, we may remark that the series in (100) and (101) could have been expressed in closed form in terms of Bessel functions of half-integral order; but this would have led back to a polynomial in ξ^{-1} which, like (97), would have been unsuitable for numerical calculation.

From (89) and (90) we have finally arrived at the asymptotic formula

$$a_{nr} \approx \frac{e_{n-r}}{4^{n-r}} \binom{2n}{2r} \theta_r. \quad (104)$$

We can also make use of Stirling's formula:

$$n! \approx n^{n+\frac{1}{2}} e^{-n} (2\pi)^{\frac{1}{2}} \left(1 + \frac{1}{12n} + \dots\right), \quad (105)$$

to get a rather less accurate but sometimes more convenient asymptotic formula

$$a_{nr} \approx \frac{4n^{2n+\frac{1}{2}} \left(1 + \frac{1}{24n}\right)}{\pi r^{2r+\frac{1}{2}} \left(1 + \frac{1}{24r}\right)} \left(\frac{\pi e}{2}\right)^{-2(n-r)} \left(1 - \frac{1}{3^{2n-2r+1}}\right) \theta_r, \quad (106)$$

by substituting (44) into (104) with the approximation $T_n = 1 - 3^{-2n-1}$. Table 4 gives the exact values of a_{nr} together with the approximations (104) and (106) for $n = 10$, and Table 5 for $n = 15$. The notation used for powers of 10 in Tables 4 and 5 is the same as in Table 3. The relatively poor performance of

Table 4. Exact and approximate values of a_{nr} when $n = 10$.

r	Exact	(104)	(106)
1	1 109 652 905	1 109 652 904:10	11 102:10
2	1 109 652 905	1 109 652 896:10	11 098:10
3	393 999 940	393 999 917: 9	39 403: 9
4	72 346 915	72 346 882: 8	72 350: 8
5	8 140 201	8 140 172: 7	81 404: 7
6	619 455	619 438: 6	61 945: 6
7	34 020	34 013: 5	3 401: 5
8	1 410	1 408: 4	1 407: 4
9	45	4 450: 2	4 423: 2
10	1	9 428: 0	8 003: 0

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again

Table 5. Exact and approximate values of a_{nr} when $n = 15$.

r	Exact	(104)	(106)
1	1 291 885 088 448 017 715	129 188 508 844 799:19	129 255:19
2	1 291 885 088 448 017 715	12 918 850 884 478:19	129 210:19
3	458 704 384 454 574 365	4 587 043 844 541:18	458 741:18
4	84 228 064 915 705 380	842 280 649 151:17	842 318:17
5	9 477 041 183 699 628	94 770 411 831:16	947 731:16
6	721 200 878 840 225	7 212 008 785:15	721 214:15
7	39 615 920 415 765	396 159 203:14	396 164:14
8	1 645 181 597 880	164 518 155:13	164 520:13
9	53 474 070 910	5 347 406:11	534 744:11
10	1 397 545 578	1 397 543:10	139 755:10
11	30 008 550	300 080: 8	300 081: 8
12	538 720	5 386: 6	53 864: 6
13	8 190	8 181: 4	8 179: 4
14	105	1 042: 3	1 036: 3
15	1	9 609: 0	8 157: 0

(106) could be improved by taking further terms in Stirling's approximation (105). However (106) comes into its own when n , r and $n-r$ are all so large that the factorials and e_{n-r} in (104) cannot be computed without the use of (44) and (105): for example using $\frac{1}{288}n^{-2}$ as the next term after $\frac{1}{12}n^{-1}$ in (105)

$$a_{1000,500} = 0.7421787178\dots \times 10^{2671} \quad (107)$$

In summary, we have proved that the numbers a_{nr} ($1 \leq r \leq n$) are all positive integers, we have got specific formulae (39) and (47) for them, together with asymptotic expressions (104) and (106). The route adopted in this paper for reaching these results was chosen principally to illustrate various manipulative techniques. It is not the only available route nor the shortest one: you may well and properly have followed a quite different course provided that it led to the desired results. The main thing is that you should have developed enough manipulative skill to get the results *somehow*.

If, following Whittaker and Watson, we define the Bernoulli numbers $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, ... by

$$\sum_{n=1}^{\infty} \frac{B_n z^{2n}}{(2n)!} = 1 - \frac{1}{2} \cot \frac{1}{2} z, \quad (108)$$

then the $b_n = a_{n1}$ in equation (6) are related by $b_n = (2^{2n+1} - 2)B_n$ because of the identity $\tan x = \cot x - 2 \cot 2x$. Professor Knuth tells me that these b_n are known as the Genocchi numbers (see Sloane (1973)). Thus Table 2 has Genocchi numbers at the top and Euler numbers at the bottom. Knuth also remarks that MACSYMA will compute the polynomials $p_n(y)$ rapidly with comparatively little memory if one uses `taylor(f(z) + f(-z), z, 0, 20)`, then `ratsubst`, then `taylor` again.

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