Adventures with the OEIS: Five sequences Tony may like

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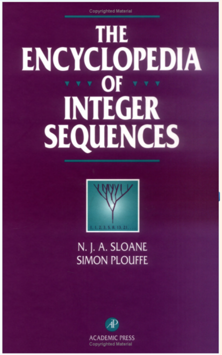
https://www.carma.newcastle.edu.au/jon/oeistalk.pdf

Outline

Introduction to Sloane's on-and-off line encyclopedia

I shall describe five encounters over nearly 30 years with Sloane's (Online) Encylopedia of Integer Sequences:



- **1973** published book (Sloane) with 2,372 entries
- **1995** published book (Sloane & Plouffe) with 5,488 entries (See SIAM Review at `https://carma.newcastle.edu.au/jon/sloane/sloane.html`.)
- **1994–1996** went on line with approximately 16,000 entries
- **Nov 15 21:28 EST 2015** has 263,957 entries – all sequences used accessed Nov 15–22

OEIS in action

https://oeis.org/

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Annual Appeal: Please make a donation (tax deductible in USA) to keep the OEIS running. Over 4500 articles have referenced us, often saying "we would not have discovered this result without the OEIS".

1,1,5,61,1385 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,1,5,61,1385**

Displaying 1-2 of 2 results found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A000364 Euler (or secant or "Zig") numbers: e.g.f. (even powers only) sech(x)=1/cosh(x). +20 148
(Formerly M4019 N1667)

1, 1, 5, 61, 1385, 50521, 2702765, 199360981, 19391512145, 2404879675441, 370371188237525, 69348874393137901, 15514534163557086905, 4087072509293123892361, 1252259641403629865468285, 441543893249023104553682821, 177519391579539289436664789665 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

COMMENTS
Inverse Gudermannian gd^(-1)(x) = log(sec(x) + tan(x)) = log(tan(Pi/4 + x/2)) = atanh(sin(x)) = 2 * atanh(tan(x/2)) = 2 * atanh(csc(x) - cot(x)). - Michael Somos, Mar 19 2011

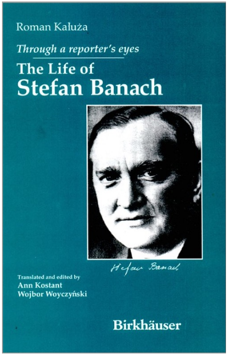
a(n) = number of downup permutations of [2n]. Example: a(2)=5 counts 4231, 4132, 3241, 3142, 2143. - David Callan, Nov 21 2011

a(n) = number of increasing full binary trees on vertices {0,1,2,...,2n} for which the leftmost leaf is labeled 2n. - David Callan, Nov 21 2011

a(n) = number of unordered increasing trees of size 2n+1 with only even degrees allowed and degree-weight generating function given by cosh(t). - Markus Kuba, Sep 13 2014

a(n) = number of standard Young tableaux of skew shape (n+1,n,n-1,...,3,2)/(n-1,n-2,...2,1). - Ran Pan, Apr 10 2015

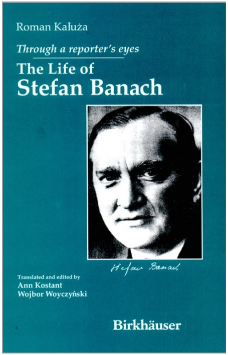
Stefan Banach (1892–1945) ... the OEIS notices analogies



A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories.

See www-history.mcs.st-andrews.ac.uk/Quotations/Banach.html

Stefan Banach (1892–1945) ... the OEIS notices analogies



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See `www-history.mcs.st-andrews.ac.uk/Quotations/Banach.html`

OEIS also now recognises numbers: 1.4331274267223117583...

Answer

- `A060997` Decimal representation of continued fraction $1, 2, 3, 4, 5, 6, 7, \ldots$ ($I_0(2)/I_1(2)$).

James Gregory (1638–1885) & Leonard Euler (1707–1783)

The sequence (A000364 (1/2))

$$2, -2, 10, -122, 2770 \ldots$$

James Gregory (1638–1885) & Leonard Euler (1707–1783)

The sequence (A000364 (1/2))

$$2, -2, 10, -122, 2770 \ldots$$

Answer

- `A011248` Twice `A000364`. Euler (or secant or "Zig") numbers: e.g.f. (even powers only) $\mathrm{sech}(x) = 1/\cosh(x)$.

James Gregory (1638–1885) & Leonard Euler (1707–1783)

The story

In 1988 Roy North observed that Gregory's series for π,

$$\pi = 4\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \qquad (1)$$

when truncated to 5,000,000 terms, gives a value differing strangely from the true value of π. *Here is the truncated Gregory value and the true value of* π:

3.14159245358979323846464338327950278419716939938730582097494182230781640
3.14159265358979323846264338327950288419716939937510582097494459230781640

Errors: 2 −2 10 −122 2770

James Gregory (1638–1885) & Leonard Euler (1707–1783)

The story

The series value differs, as one might expect from a series truncated to 5,000,000 terms, in the seventh decimal place—a "4" where there should be a "6." But the next 13 digits are correct!

Then, following another erroneous digit, the sequence is once again correct for an additional 12 digits. In fact, of the first 46 digits, only four differ from the corresponding decimal digits of π.

Further, the "error" digits appear to occur in positions that have a period of 14, as shown above.

James Gregory (1638–1885) & Leonard Euler (1707–1783)

The story

We note that each integer is even; dividing by two, we obtain $(1, -1, 5, -122, 1385)$. *Sloane has told us we have the Euler numbers defined in terms of Taylor's series for* $\sec x$*:*

$$\sec x = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k} x^{2k}}{(2k)!}. \tag{2}$$

Indeed, we see the asymptotic expansion base 10 on the screen:

$$\frac{\pi}{2} - 2\sum_{k=1}^{N/2} \frac{(-1)^{k+1}}{2k-1} \approx \sum_{m=0}^{\infty} \frac{E_{2m}}{N^{2m+1}} \tag{3}$$

This works in hex (!!) and $\log 2$ *yields the tangent numbers.*

James Gregory (1638–1885) & Leonard Euler (1707–1783)

Nico Temme's 1995 Wiley book *Special Functions: An Introduction to the Classical Functions of Mathematical Physics* starts with this example.

References

- J.M. Borwein, P.B. Borwein, and K. Dilcher, "Euler numbers, asymptotic expansions and pi," *MAA Monthly*, **96** (1989), 681–687.
- See also *Mathematics by Experiment* §2.10 and "I prefer Pi" in MAA Monthly, March 2015.

Siméon Poisson (1781–1840) & ET Bell (1883–1960)

The sequence (A000110 (1/10))

$$1, 1, 2, 5, 15, 52, 203, 877, 4140 \ldots$$

Siméon Poisson (1781–1840) & ET Bell (1883–1960)

The sequence (A000110 (1/10))

$$1, 1, 2, 5, 15, 52, 203, 877, 4140 \ldots$$

Answer

- Bell or exponential numbers: number of ways to partition a set of n labeled elements.

Siméon Poisson (1781–1840) & ET Bell (1883–1960)

The story

MAA Unsolved Problem: *For $t > 0$, let*

$$m_n(t) = \sum_{k=0}^{\infty} k^n \exp(-t) \frac{t^k}{k!}$$

be the n-th moment of a Poisson distribution *with parameter t. Let $c_n(t) = m_n(t)/n!$. Show*

(a) $\{m_n(t)\}_{n=0}^{\infty}$ *is log-convex for all $t > 0$.*

(b) $\{c_n(t)\}_{n=0}^{\infty}$ *is not log-concave for $t < 1$.*

(c*) $\{c_n(t)\}_{n=0}^{\infty}$ *is log-concave for $t \geq 1$.*

Siméon Poisson (1781–1840) & ET Bell (1883–1960)

The story

(b) As

$$m_{n+1}(t) = t \sum_{k=0}^{\infty} (k+1)^n \exp(-t) \frac{t^k}{k!},$$

on applying the binomial theorem to $(k+1)^n$, we see that

$$m_{n+1}(t) = t \sum_{k=0}^{n} \binom{n}{k} m_k(t), \qquad m_0(t) = 1.$$

In particular for $t = 1$, we obtain the sequence

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, \ldots$$

These we have learned are the Bell numbers.

Siméon Poisson (1781–1840) & ET Bell (1883–1960)

The story

OEIS `A001861` *also tell us that for $t = 2$, we have generalized Bell numbers, and gives us the exponential generating functions.*

The Bell numbers were known earlier to Ramanujan.

Now an explicit computation shows that

$$t\,\frac{1+t}{2} = c_0(t)\,c_2(t) \leq c_1(t)^2 = t^2$$

exactly if $t \geq 1$. Also, preparatory to the next part, a simple calculation shows that

$$\sum_{n\geq 0} c_n u^n = \exp\left(t(e^u - 1)\right). \tag{4}$$

Siméon Poisson (1781–1840) & ET Bell (1883–1960)

The story

*(c^*) (The * indicates this was unsolved.) We appeal to a then recent theorem due to Canfield. A search in 2001 on* MathSciNet *for "Bell numbers" since 1995 turned up 18 items. Canfield showed up as paper #10. Later,* Google *found the paper immediately!*

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Theorem (Canfield)

If a sequence $1, b_1, b_2, \cdots$ is non-negative and log-concave, then so is $1, c_1, c_2, \cdots$ determined by the generating function equation

$$\sum_{n\geq 0} c_n u^n = \exp\left(\sum_{j\geq 1} b_j \frac{u^j}{j}\right).$$

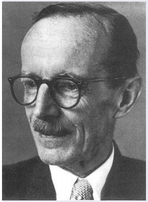
Siméon Poisson (1781–1840) & ET Bell (1883–1960)

References

- *Experimentation in Mathematics* §1.11.
- E.A. Bender and R.E. Canfield, "Log-concavity and related properties of the cycle index polynomials," *J. Combin. Theory Ser. A* **74** (1996), 57–70.
- Solution to Unsolved Problem 10738, posed by Radu Theodorescu in the 1999 *American Mathematical Monthly*.

Erwin Madelung (1881–1972) & Richard Crandall (1947–2012)

The sequence (A055745 (1/3))



$$1, 2, 6, 10, 22, 30, 42, 58, 70, 78, 102, 130$$
$$190, 210, 330, 462 \ldots$$

Erwin Madelung (1881–1972) & Richard Crandall (1947–2012)

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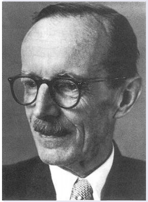

$1, 2, 6, 10, 22, 30, 42, 58, 70, 78, 102, 130$
$190, 210, 330, 462 \ldots$

Answer

- Squarefree numbers not of form $ab + bc + ca$ for $1 \leq a \leq b \leq c$ (probably the list is complete).
- A034168 Disjoint discriminants (one form per genus) of type 2 (doubled).

Erwin Madelung (1881–1972) & Richard Crandall (1947–2012)

The story

A lovely 1986 formula for $\theta_4^3(q)$ due to Andrews is

$$\theta_4^3(q) = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+q^n} - 2\sum_{n=1,|j|<n}^{\infty} (-1)^j q^{n^2-j^2} \frac{1-q^n}{1+q^n}. \tag{5}$$

From (5) *Crandall obtains*

$$\sum_{n,m,p>0}^{\infty} \frac{(-1)^{n+m+p}}{(n^2+m^2+p^2)^s} = -4\sum_{n,m,p>0}^{\infty} \frac{(-1)^{n+m+p}}{(nm+mp+pn)^s} - 6\alpha^2(s). \tag{6}$$

Here $\alpha(s) = \left(1-2^{1-s}\right)\zeta(s)$ is the alternating zeta function.

Erwin Madelung (1881–1972) & Richard Crandall (1947–2012)

The story

Crandall used Andrew's formula (6) to find a new representation for Madelung's constant

$$M_3(2s) := \sum_{n,m,p>0}^{\infty} \frac{(-1)^{n+m+p}}{(n^2+m^2+p^2)^s}.$$

He then asked me what numbers were not of the form

$$ab + bc + ca.$$

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It was bed-time in Vancouver so I asked my ex-PDF Roland Girgensohn in Munich.

Erwin Madelung (1881–1972) & Richard Crandall (1947–2012)

The story

- *When I woke up, Roland had used* MATLAB *to send all* 18 *solutions up to* $50,000$.
- *Also* $4, 18$ *are the only non-square free solutions.*
- *I recognised the square-free numbers as singular values of type II (Dickson)*

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The story

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- *Also* $4, 18$ *are the only non-square free solutions.*
- *I recognised the square-free numbers as singular values of type II (Dickson)*
- *One more* 19*-th solution* $s > 10^{11}$ *might exist but only without GRH.*

Erwin Madelung (1881–1972) & Richard Crandall (1947–2012)

The story (The Newcastle connection)

. . . Born decided to investigate the simple ionic crystal-rock salt (sodium chloride) – using a ring model. He asked Lande to collaborate with him in calculating the forces between the lattice points that would determine the structure and stability of the crystal. Try as they might, the mathematical expression that Born and Lande derived contained a summation of terms that would not converge. Sitting across from Born and watching his frustration, Madelung offered a solution. His interest in the problem stemmed from his own research in Goettingen on lattice energies that, six years earlier, had been a catalyst for Born and von Karman's article on specific heat.

Erwin Madelung (1881–1972) & Richard Crandall (1947–2012)

The story (The Newcastle connection)

The new mathematical method he provided for convergence allowed Born and Lande to calculate the electrostatic energy between neighboring atoms (a value now known as the Madelung constant). Their result for lattice constants of ionic solids made up of light metal halides (such as sodium and potassium chloride), and the compressibility of these crystals agreed with experimental results.

Max Born was singer and actress Olivia Newton-John's maternal grandfather. Actually, soon after they discovered they had forgotten to divide by two in the compressibility analysis. This ultimately led to the abandonment of the Bohr-Sommerfeld planar model of the atom.

Erwin Madelung (1881–1972) & Richard Crandall (1947–2012)

Ignorance can be bliss

Luckily, we only looked at the OEIS *after* the paper was written.

References

- Jonathan Borwein and Kwok-Kwong Stephen Choi, "On the representations of $xy + yz + zx$," *Experimental Mathematics*, **9** (2000), 153–158.
- J. Borwein, L. Glasser, R. McPhedran, J. Wan, and J. Zucker, *Lattice Sums: Then and Now.* Encyclopedia of Mathematics and its Applications, **150**, Cambridge University Press, 2013.

Cyril Domb (1920–2012) & Karl Pearson (1857–1936)

The sequence (A002895 & A253095)

$$1, 4, 28, 256, 2716, 31504, 387136, 4951552 \ldots$$

and

$$1, 4, 22, 148, 1144, 9784, 90346, 885868, 9115276 \ldots$$

Cyril Domb (1920–2012) & Karl Pearson (1857–1936)

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$$1, 4, 28, 256, 2716, 31504, 387136, 4951552 \ldots$$

and

$$1, 4, 22, 148, 1144, 9784, 90346, 885868, 9115276 \ldots$$

Answers

- Domb numbers: number of $2n$-step polygons on diamond lattice.
- Moments of 4-step random walk in 2 and 4 dimensions.

Cyril Domb (1920–2012) & Karl Pearson (1857–1936)

The story

We developed the following expression for the even moments. It is only entirely integer for $d = 2, 4$.
In two dimensions it counts abelian squares.
What does it count in four space?

Theorem (Multinomial sum for the moments)

The even moments of an n-step random walk in dimension $d = 2\nu + 2$ are given by

$$W_n(\nu; 2k) = \frac{(k+\nu)!\nu!^{n-1}}{(k+n\nu)!} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1, \dots, k_n} \binom{k+n\nu}{k_1+\nu, \dots, k_n+\nu}.$$

Cyril Domb (1920–2012) & Karl Pearson (1857–1936)

The story (Generating function for 3 steps in 4 dimensions)

For $d = 4$, so $\nu = 1$, the moments are sequence A103370. The OEIS also records a hypergeometric form of the generating function (as the linear combination of a hypergeometric function and its derivative), added by Mark van Hoeij.

On using linear transformations of hypergeometric functions, we have more simply that

$$\sum_{k=0}^{\infty} W_3(1;2k)x^k = \frac{1}{2x^2} - \frac{1}{x} - \frac{(1-x)^2}{2x^2(1+3x)} {}_2F_1\left(\left.\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 2\end{matrix}\right| \frac{27x(1-x)^2}{(1+3x)^3}\right),$$

which we are able to generalise (the planar o.g.f has the same "form") – note the Laurent polynomial.

Cyril Domb (1920–2012) & Karl Pearson (1857–1936)

Theorem (Generating function for even moments with three steps)

For integers $\nu \geq 0$ *and* $|x| < 1/9$, *we have*

$$\sum_{k=0}^{\infty} W_3(\nu; 2k) x^k = \frac{(-1)^\nu}{\binom{2\nu}{\nu}} \frac{(1-1/x)^{2\nu}}{1+3x} {}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1+\nu \end{matrix} \middle| \frac{27x(1-x)^2}{(1+3x)^3} \right) - q_\nu\left(\frac{1}{x}\right), \tag{7}$$

where $q_\nu(x)$ *is a polynomial (that is,* $q_\nu(1/x)$ *is the principal part of the hypergeometric term on the right-hand side).*

Cyril Domb (1920–2012) & Karl Pearson (1857–1936)

Theorem (Generating function for even moments with three steps)

For integers $\nu \geq 0$ and $|x| < 1/9$, we have

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where $q_\nu(x)$ is a polynomial (that is, $q_\nu(1/x)$ is the principal part of the hypergeometric term on the right-hand side).

$$\sum_{k=0}^{\infty} W_3(0; 2k)x^k = \frac{1}{1+3x} {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3} \\ 1\end{matrix}\middle| \frac{27x(1-x)^2}{(1+3x)^3}\right).$$

Cyril Domb (1920–2012) & Karl Pearson (1857–1936)

Reference

- J.M. Borwein, A. Straub and C. Vignat, "Densities of short uniform random walks in higher dimensions," preprint, 2015. See `http://www.carma.newcastle.edu.au/jon/dwalks.pdf`.
- J. Borwein, A. Straub, J. Wan and W. Zudilin, with an Appendix by Don Zagier, "Densities of short uniform random walks," *Canadian. J. Math.* **64** (5), (2012), 961–990.

Poisson & Crandall

The sequence (A218147)

$$2, 2, 4, 4, 12, 8, 18, 8, 30, 16, 36, 24, 32, 32, 64, 36, 90, 32, 96, 60, 132, 64, 100, 72 \ldots$$

Poisson & Crandall

The sequence (A218147)

$2, 2, 4, 4, 12, 8, 18, 8, 30, 16, 36, 24, 32, 32, 64, 36, 90, 32, 96, 60, 132, 64, 100, 72 \ldots$

Answer

- Conjectured degree of polynomial satisfied by

$$m(n) := \exp(8\pi\phi_2(1/n, 1/n).$$

- `A079458`: $4\,m(n)$ is the number of Gaussian integers in a reduced system modulo n.

Poisson & Crandall

The story

$$\phi_2(x,y) = \frac{1}{\pi^2} \sum_{m,n \text{ odd}} \frac{\cos(m\pi x)\cos(n\pi y)}{m^2+n^2}. \tag{8}$$

Crandall conjectured and I then proved that when x, y are rational

$$\phi_2(x,y) = \frac{1}{\pi} \log A, \tag{9}$$

where A is algebraic. Both computation and proof exploited:

$$\phi_2(x,y) = \frac{1}{2\pi} \log \left| \frac{\theta_2(z,q)\theta_4(z,q)}{\theta_1(z,q)\theta_3(z,q)} \right|, \tag{10}$$

where $q = e^{-\pi}$ and $z = \frac{\pi}{2}(y + ix)$.

Poisson & Crandall

The story

2012 *Jason Kimberley (UofN) remarkably, observed degree* $m(s)$ *of minimal polynomial for* $x = y = 1/s$ *is as follows. Set* $m(2) = 1/2$. *For primes* p *congruent to 1 mod 4, set* $m(p) = \mathrm{int}^2(p/2)$, *where* int *denotes greatest integer, and for* p *congruent to 3 mod 4, set* $m(p) = \mathrm{int}\,(p/2)(\mathrm{int}\,(p/2)+1)$. *Then with prime factorisation* $s = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$,

$$m(s) \stackrel{?}{=} 4^{r-1} \prod_{i=1}^{r} p_i^{2(e_i-1)}\, m(p_i). \tag{11}$$

- **2015** (11) *holds for all tested cases where* s *now ranges up to 50 – save* $s = 41, 43, 47, 49$, *which are still too costly to test.*
- *JK conjectured closed form of polynomials – proven by WL!*

Searching for 387221579866, from P_{11}, we learn that Gordan Savin and David Quarfoot (2010) define a sequence of polynomials $\psi_s(x, y)$ with $\psi_0 = \psi_1 = 1$ while $\psi_2 = 2y, \psi_3 = 3x^4 + 6x^2 + 1$, $\psi_4 = 2y(2x^6 + 10x^4 - 10x^2 - 2)$ and

$$\psi_{2n+1} = \psi_{n+2}\psi_n^3 - \psi_{n-1}\psi_{n+1}^3 \qquad (n \geq 2) \qquad (12)$$

$$2y\psi_{2n} = \psi_n\left(\psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2\right) \qquad (n \geq 3). \qquad (13)$$

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Conjecture (Kimberley)

(a) For each integer $s \geq 1$, $P_s(-x^2)$ is a prime factor of $\psi_s(x)$. In fact, it is the unique prime factor of degree $2 \times A218147(s)$.
(b) (divisibility) $m \mid n$ implies $\psi_m \mid \psi_n$ (confirmed for $n \leq 120$).

- For primes of form $4n+3$, $\psi_s(x)$ is irreducible over $Q(i)$.
- Conjecture (a) confirmed for $s = 52$.

Poisson & Crandall ... visualising big data

Table: 192-degree minimal polynomial with 85 digit coefficients found by multipair PSLQ for the case $x = y = 1/35$.

Poisson & Crandall for aberration correction

References

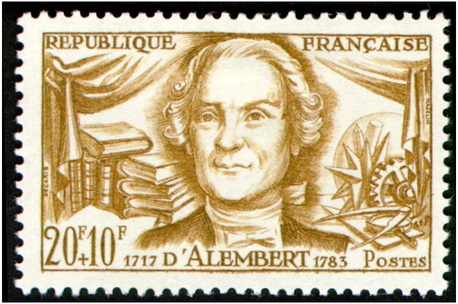
- D.H. Bailey, J.M. Borwein, R.E. Crandall, I.J. Zucker, "Lattice sums arising from the Poisson equation." *Journal of Physics A*, **46** (2013) #115201 (31pp).
- D.H. Bailey, J.M. Borwein, J. Kimberley, "Discovery and computation of large Poisson polynomials. With an appendix by Watson Ladd." To appear *Experimental Math*, 2016.
- G. Savin, D. Quarfoot, "On attaching coordinates of Gaussian prime torsion points of $y^2 = x^3 + x$ to $Q(i)$," March 2010. `www.math.utah.edu/~savin/EllipticCurvesPaper.pdf`

Conclusion

- OEIS is an amazing *instrumental* resource (See 2015 interview in Quanta)
    - `https://www.quantamagazine.org/20150806-neil-sloane-oeis-interview/`
- A model both for curation and for moderation
    - with other resources such as email-based super seeker
- As with all tools, the OEIS can help (very often) and it can hinder (much less often)
- Coming soon: J. Monaghan, L. Troché and JMB, *Tools and Mathematics*, Springer (Mathematical Education), 2015.

Happy Seventy, Tony

Algebra is generous; she often gives more than is asked of her. *(Jean d´Alembert, 1717–1783)*



This Guy may be you in 29 years.