LINEAR DIVISIBILITY SEQUENCES AND CHEBYSHEV POLYNOMIALS

PETER BALA, March 2014

Williams and Guy extended Lucas' two parameter family of second-order linear divisibility sequences to a three parameter family of fourth-order linear divisibility sequences. We give a formula using (bivariate) Chebyshev polynomials for the terms of the Williams and Guy sequences. We briefly consider Chebyshev analogs of two other families of linear divisibility sequences.

1. Introduction. A sequence $\{a(n)\}_{n\geq 1}$ of elements of an integral domain D is a divisibility sequence if $a(n)$ divides $a(m)$ whenever n divides m and $a(n) \neq 0$. We call $\{a(n)\}$ a linear divisibility sequence of order k if the sequence also satisfies a homogeneous linear recurrence of order k with coefficients from the domain D.

If P_1 and P_2 are a pair of nonzero integers, the sequence

$$U_0 = 0, \quad U_1 = 1, \quad U_n = P_1 U_{n-1} - P_2 U_{n-2}, \quad n \geq 2, \tag{1}$$

is called a Lucas sequence. Lucas [3] proved that each sequence $\{U_n\}_{n\geq 1}$ is a second-order integer linear divisibility sequence. An explicit formula is

$$U_n = U_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 0, \tag{2}$$

where α and β are the zeros of the associated quadratic $x^2 - P_1 x + P_2$, so that

$$\begin{aligned} \alpha + \beta &= P_1 \\ \alpha\beta &= P_2 \end{aligned} \tag{3}$$

(we assume $\alpha - \beta \neq 0$, or equivalently $P_1^2 \neq 4P_2$).

The generating function of the Lucas sequence U_n is readily calculated as

$$\sum_{n\geq 1} U_n z^n = \frac{z}{1 - P_1 z + P_2 z^2}. \tag{4}$$

Williams and Guy [6] extended the results of Lucas and found a family of fourth-order linear divisibility sequences $W_n = W_n(P_1, P_2, Q)$, depending on the parameters P_1, P_2 and a third integer parameter Q, and having the generating function

$$\sum_{n\geq 1} W_n z^n = \frac{z(1 - Qz^2)}{1 - P_1 z + (P_2 + 2Q)z^2 - P_1 Q z^3 + Q^2 z^4}. \tag{5}$$

Our purpose in this note is to prove the following expression for W_n, analogous to equation (2), but involving Chebyshev polynomials rather than monomial polynomials:

$$W_n = \frac{t_n(\alpha, Q) - t_n(\beta, Q)}{\alpha - \beta}, \tag{6}$$

where $t_n(x,s)$ denotes the monic bivariate Chebyshev polynomial of the first kind. When $Q=0$, the generating function (5) reduces to (4), and (6) becomes (2).

2. Bivariate Chebyshev polynomials. We recall some well-known results about Chebyshev polynomials - see, for example, [4].
The classical Chebyshev polynomials of the first kind $T_n(x)$ satisfy the second-order linear recurrence

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \tag{7}$$

with the starting values $T_0(x) = 1$ and $T_1(x) = x$.
They are given by the explicit formula

$$T_n(x) = \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2} \tag{8}$$

and have the generating function

$$\sum_{n\geq 0} T_n(x)z^n = \frac{1-xz}{1-2xz+z^2}\ . \tag{9}$$

The Chebyshev polynomials $T_n(x)$ satisfy the composition rule

$$T_n(T_m(x)) = T_{nm}(x), \quad n,m \geq 0, \tag{10}$$

which is also satisfied by the sequence of monomial polynomials $\{x^n\}$. As we will see later, it is this composition property that allows us to construct divisibility sequences.
In order to obtain the full 3-parameter family of divisibility sequences found by Williams and Guy, it turns out we need to work with two variable Chebyshev polynomials $t_n(x,s)$ defined as follows. Let $s \neq 0$ be a complex parameter. The bivariate Chebyshev polynomials $t_n(x,s)$ of the first kind are defined by

$$t_n(x,s) = 2(\sqrt{s})^n T_n\left(\frac{x}{2\sqrt{s}}\right) \tag{11}$$

(the factors of 2 are included to ensure $t_n(x,s)$ is a monic polynomial in the variable x). The first few values are

$$\begin{aligned} t_0(x) &= 2 & t_3(x) &= x^3 - 3sx \\ t_1(x) &= x & t_4(x) &= x^4 - 4sx^2 + 2s^2 \\ t_2(x) &= x^2 - 2s & t_5(x) &= x^5 - 5sx^3 + 5s^2x. \end{aligned}$$

The polynomials $t_n(x,s)$ appear in the literature under a variety of names. Bircan et al. [1] call them adapted Chebyshev polynomials. They are also known as the Dickson polynomials of the first kind [7], usually denoted by $D_n(x,s)$.
The following properties of the bivariate Chebyshev polynomials are easily derived from the corresponding properties of the classical Chebyshev polynomials.

There is the explicit formula

$$t_n(x,s) = \left(\frac{x+\sqrt{x^2-4s}}{2}\right)^n + \left(\frac{x-\sqrt{x^2-4s}}{2}\right)^n. \tag{12}$$

The generating function is

$$\sum_{n\geq 0} t_n(x,s)z^n = \frac{2-xz}{1-xz+sz^2}, \tag{13}$$

from which we get the second-order linear recurrence

$$t_n(x,s) = xt_{n-1}(x,s) - st_{n-2}(x,s) \tag{14}$$

with starting values $t_0(x,s) = 2, t_1(x,s) = x$. Therefore, $t_n(x,s) \in \mathbb{Z}[x,s]$. In particular, from (14) we have $t_n(x,0) = x^n$ for $n \geq 1$, so the bivariate Chebyshev polynomials may be viewed as generalizations of the monomial polynomials.
Using (10), we find the composition rule for the bivariate Chebyshev polynomials takes the form

$$t_m(t_n(x,s), s^n) = t_{nm}(x,s), \quad n,m \geq 0. \tag{15}$$

We are now ready to construct some divisibility sequences.

Proposition 2.1 Let Q be an integer. The sequence of bivariate polynomials $\{W_n(x,y)\}_{n\geq 1}$ defined by

$$W_n(x,y) = \frac{t_n(x,Q) - t_n(y,Q)}{x-y} \tag{16}$$

is a fourth-order linear divisibility sequence in the domain $\mathbb{Z}[x,y]$.
Proof
Firstly, we show $W_n(x,y)$ is a polynomial in the ring $\mathbb{Z}[x,y]$ for each natural number n.
The elementary identity

$$\frac{x^n-y^n}{x-y} = x^{(n-1)} + x^{(n-2)}(-y) + \cdots + (-y)^{(n-1)}, \quad n = 1,2,3,... \tag{17}$$

tells us that the polynomial $x-y$ divides the polynomial $x^n - y^n$ in the ring $\mathbb{Z}[x,y]$. Since Q is an integer we have $t_n(x,Q) \in \mathbb{Z}[x]$. It follows from (17) that $x-y$ divides the polynomial $t_n(x,Q) - t_n(y,Q)$ in the ring $\mathbb{Z}[x,y]$, and hence $W_n(x,y) \in \mathbb{Z}[x,y]$.
To show the polynomial $W_n(x,y)$ divides the polynomial $W_{nm}(x,y)$ for all natural numbers n and m we use the composition rule (15) to find

$$\begin{aligned} \frac{W_{nm}(x,y)}{W_n(x,y)} &= \frac{t_{nm}(x,Q) - t_{nm}(y,Q)}{t_n(x,Q) - t_n(y,Q)} \\ &= \frac{t_m(t_n(x,Q),Q^n) - t_m(t_n(y,Q),Q^n)}{t_n(x,Q) - t_n(y,Q)} \\ &= \frac{t_m(X,Q^n) - t_m(Y,Q^n)}{X-Y} \end{aligned}$$

where $X = t_n(x,Q) \in \mathbb{Z}[x]$ and $Y = t_n(y,Q) \in \mathbb{Z}[y]$. Thus, by the first part of the proof, $W_{nm}(x,y)/W_n(x,y)$ is a polynomial in $\mathbb{Z}[X,Y]$, and is therefore a polynomial in $\mathbb{Z}[x,y]$. Hence $W_n(x,y)$ divides $W_{nm}(x,y)$ in the ring $\mathbb{Z}[x,y]$.
Using (13), we calculate the generating function

$$\begin{aligned}\sum_{n\geq 1} W_n(x,y)z^n &= \frac{1}{x-y}\left(\frac{2-xz}{1-xz+Qz^2} - \frac{2-yz}{1-yz+Qz^2}\right)\\ &= \frac{z(1-Qz^2)}{(1-xz+Qz^2)(1-yz+Qz^2)}\\ &= \frac{z(1-Qz^2)}{1-(x+y)z+(xy+2Q)z^2-(x+y)Qz+Q^2z^4}\,. \qquad (18)\end{aligned}$$

From the form of the denominator in (18) we see that the polynomial $W_n = W_n(x,y)$ satisfies the fourth-order linear recurrence

$$W_n = (x+y)W_{n-1} - (xy+2Q)W_{n-2} + (x+y)QW_{n-3} - Q^2W_{n-4}, \quad n \geq 4. \qquad (19)$$

□

Clearly, we can get linear divisibilty sequences of integers from the sequence of polynomials $W_n(x,y)$ in Proposition 2.1 by suitably specializing x and y, for example, by taking x and y to be distinct integers. In fact, a wider class of integer divisibility sequences is possible. Observe that the polynomial $W_n(x,y)$ is symmetric in x and y and hence, by the fundamental theorem of symmetric polynomials, can be written as a polynomial with integer coefficients in the elementary symmetric polynomials $x+y$ and xy. The same holds true for the symmetric polynomials $W_{nm}(x,y)/W_n(x,y)$ for all natural numbers n and m. Therefore, in order for $\{W_n(x,y)\}$ to be an integer divisibility sequence, it suffices to choose values for x and y so that both $x+y$ and xy are integers. Accordingly, let P_1 and P_2 be a pair of nonzero integers and define complex numbers α and β by

$$\begin{aligned}\alpha+\beta &= P_1\\ \alpha\beta &= P_2, \end{aligned} \qquad (20)$$

so that α,β are the roots of the quadratic equation $x^2 - P_1x + P_2 = 0$. We suppose further that $\alpha - \beta \neq 0$, or equivalently $P_1^2 \neq 4P_2$. Then it follows from Proposition 2.1 and the preceeding remarks that

$$W_n(\alpha,\beta) = \frac{t_n(\alpha,Q) - t_n(\beta,Q)}{\alpha-\beta}, \quad n \geq 1, \qquad (21)$$

is a well-defined fourth-order linear divisibility sequence of integers, depending on the 3 integer parameters P_1, P_2 and Q.
Substituting $x = \alpha$ and $y = \beta$ in (18) we obtain the generating function

$$\sum_{n\geq 1} W_n(\alpha,\beta)z^n = \frac{z(1-Qz^2)}{1-P_1z+(P_2+2Q)z^2-P_1Qz^3+Q^2z^4}\,. \qquad (22)$$

This is the same as (5), the generating function for Williams and Guy's 3-parameter divisibility sequence. Thus, we have established the following result: Williams and Guy's fourth-order linear divisibility sequence $W_n = W_n(P_1, P_2, Q)$, with integer parameters P_1, P_2 and Q, is given by the formula

$$W_n = \frac{t_n(\alpha, Q) - t_n(\beta, Q)}{\alpha - \beta}, \quad n \geq 1,$$

where

$$\begin{aligned} \alpha + \beta &= P_1 \\ \alpha\beta &= P_2 \end{aligned}$$

and where $t_n(x, Q)$ denotes the monic bivariate Chebyshev polynomial of the first kind with parameter Q.
The recurrence equation for the sequence W_n follows from (19)

$$W_n = P_1 W - (P_2 + 2Q)W_{n-2} + P_1 Q W_{n-3} - Q^2 W_{n-4}, \quad n \geq 4. \tag{23}$$

3. The 2x2 matrix approach. There is a well-known connection between 2x2 matrices and Lucas sequences. Let P_1 and P_2 be a pair of nonzero integers and let A be a 2x2 matrix

$$\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $\mathrm{trace(A)} = P_1$ and $\det(\mathrm{A}) = P_2$.
It is not difficult to show that the non-diagonal elements of the matrix power A^n are multiples of the Lucas sequence $U_n(P_1, P_2)$:

$$\mathrm{A}^n = \begin{bmatrix} * & bU_n(P_1, P_2) \\ cU_n(P_1, P_2) & * \end{bmatrix}. \tag{24}$$

Similarly, it can be shown that the non-diagonal elements of the 2x2 matrix $t_n(\mathrm{A}, Q)$, $Q \in \mathbb{Z}$, are multiples of the Williams and Guy sequence $W_n(P_1, P_2, Q)$:

$$t_n(\mathrm{A}, Q) = \begin{bmatrix} * & bW_n(P_1, P_2, Q) \\ cW_n(P_1, P_2, Q) & * \end{bmatrix}. \tag{25}$$

Sketch proof Let $\mathrm{X} \in \mathrm{GL}(2, \mathbb{C})$ be a 2x2 matrix. One shows that the 2x2 matrices $t_n(\mathrm{X}) = t_n(\mathrm{X}, s)$ satisfy the fourth-order linear recurrence

$$t_n(\mathrm{X}) = \mathrm{trace}\ (\mathrm{X}) t_{n-1}(\mathrm{X}) - (\det\ (\mathrm{X}) + 2s) t_{n-2}(\mathrm{X}) + s.\mathrm{trace}\ (\mathrm{X}) t_{n-3}(\mathrm{X}) - s^2 t_{n-4}(\mathrm{X}) \tag{26}$$

The proof is an easy consequence of the Cayley-Hamilton theorem for X and the second-order recurrence equation (14) for the bivariate Chebyshev polynomials $t_n(x, s)$. For the choices $X = A$ and $s = Q$, equation (26) has the same form as the recurrence equation (23) for the Williams and Guy sequence W_n. Consequently, each element of the array $t_n(A) = t_n(A, Q)$ satisifies the recurrence

(23). Thus to conclude that the non-diagonal elements of $t_n(A)$ are multiples of W_n it is only necessary to check the result for first few values of n.

4. Higher order divisibilty sequences. We conclude by using Chebyshev polynomials to generalize some divisibility sequences of Lehmer and Roettger.

(a) Lehmer sequences

Let P_1 and P_2 be a pair of nonzero integers. Lehmer [2] extended the Lucas sequence $U_n(P_1, P_2)$ to a fourth-order linear divisibilty sequence $L_n(P_1, P_2)$ as follows. Let α and β be the roots of the quadratic equation $x^2 - \sqrt{P_1}x + P_2 = 0$, so that

$$\begin{aligned} \alpha + \beta &= \sqrt{P_1} \\ \alpha\beta &= P_2\,. \end{aligned} \tag{27}$$

We assume $\alpha - \beta \neq 0$, or equivalently $P_1 \neq 4P_2$. The Lehmer sequence $L_n = L_n(P_1, P_2)$ is defined as

$$L_n = \begin{cases} \frac{\alpha^n - \beta^n}{\alpha - \beta} & n \text{ odd} \\ \\ \frac{\alpha^n - \beta^n}{\alpha^2 - \beta^2} & n \text{ even} \end{cases} \tag{28}$$

L_n is an integer linear divisibility sequence of order 4.

Using the bivariate Chebyshev polynomials $t_n(x, s)$ in place of monomials in (28) leads to the Chebyshev analog $\widetilde{L}_n = \widetilde{L}_n(P_1, P_2, Q)$ of Lehmer's sequence defined as

$$\widetilde{L}_n = \begin{cases} \frac{t_n(\alpha,Q) - t_n(\beta,Q)}{\alpha - \beta} & n \text{ odd} \\ \\ \frac{t_n(\alpha,Q) - t_n(\beta,Q)}{\alpha^2 - \beta^2} & n \text{ even} \end{cases} \tag{29}$$

Here Q is an integer parameter. It is not difficult to show that the sequence $\widetilde{L}_n$ is an integer divisibility sequence. By a generating function calculation we find the sequence $\widetilde{L}_n$ satisfies a linear recurrence of order 8.

(b) Roettger's cubic generalization of the Lucas sequence U_n.

Roettger [5, Chapter 3] has made a detailed study of a 3-parameter family $C_n(P_1, P_2, P_3)$ of integer linear divisibility sequences of order 6 defined as follows. Let P_1, P_2 and P_3 be integers, and let α, β, γ be the zeros of $x^3 - P_1x^2 + P_2x - P_3$. Then Roettger's sequence $C_n = C_n(P_1, P_2, P_3)$ is given by

$$C_n = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\left(\frac{\beta^n - \gamma^n}{\beta - \gamma}\right)\left(\frac{\gamma^n - \alpha^n}{\gamma - \alpha}\right). \tag{30}$$

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Let Q be an integer. We define the Chebyshev analog $\widetilde{C}_n = \widetilde{C}_n(P_1, P_2, P_3, Q)$ of Roettger's sequence by

$$\widetilde{C}_n = \left(\frac{t_n(\alpha, Q) - t_n(\beta, Q)}{\alpha - \beta}\right)\left(\frac{t_n(\beta, Q) - t_n(\gamma, Q)}{\beta - \gamma}\right)\left(\frac{t_n(\gamma, Q) - t_n(\alpha, Q)}{\gamma - \alpha}\right) \tag{31}$$

It is not difficult to show that the sequence $\widetilde{C}_n$ is an integer divisibility sequence. As an example, let $P_1 = -2, P_2 = 2, P_3 = -1$ and $Q = 1$, so the associated cubic is x^3+2x^2+2x+1 with zeros $\alpha = -1, \beta = (-1+\sqrt{-3})/2$ and $\gamma = (-1-\sqrt{-3})/2$. The sequence $\widetilde{C}_n$ begins $[1, -3, -21, 195, 244, -2835, 463, 34125, -68229, -363072, ...]$. By calculating the generating function of the sequence we find $\widetilde{C}_n$ satisfies a linear recurrence of order 24.

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