Concise Representations and Complexity of Combinatorial Assignment Problems

Extended Abstract

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ABSTRACT

We consider the computational problem of partitioning items into bundles among alternatives to maximize social welfare. Unfortunately, many important classes of this problem are computationally hard, including well-known instances in the multi-agent systems literature. In this paper we analyze novel concise representations and restrictions that admit polynomial-time algorithms for many such combinatorial assignment problems, and prove several complexity results for them. We provide efficient approximation algorithms and non-trivial exponential-time algorithms for the hard cases.

KEYWORDS

Combinatorial Assignment; Complexity; Resource Allocation; Coalition Structure Generation; Combinatorial Auctions

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1 INTRODUCTION

Efficiently distributing resources in a sustainable and effective way is one of the most important and challenging problems in society. Different *combinatorial assignment problems*, in which the goal is to divide a number of indivisible items among alternatives to maximize welfare/value, appear in a wide range of settings. Such settings include operations research, economics and artificial intelligence, with real-world applications in combinatorial auctions [27], multi-target tracking and sensor fusion [8], course assignment [4, 19], resource allocation [15], and team/coalition formation [16, 21].

Many important combinatorial assignment problems are however computationally hard, hard to approximate in polynomial time, and/or require an input size that is exponential in one of the parameters. Important examples of such problems include *coalition structure generation* [26], *combinatorial auction winner determination* [25], and *generalized assignment* [5].

In light of this, we investigate the computational aspects of two hard general classes of combinatorial assignment problems:

(1) *Utilitarian combinatorial assignment* (UCA), in which the sum of all the bundle-to-alternative assignments is maximized with the goal to maximize a system's total output. Examples include multi-agent scenarios where the goal is to simultaneously form agent teams and assign them to tasks.
(2) *Egalitarian combinatorial assignment* (ECA), in which the value of the bundle that is worst off is maximized; i.e., we want to keep the weakest link as strong as possible. An example of this is equitable distribution of indivisible resources.

To this end, our main contributions can be summarized as follows:

- We develop a novel hypergraph characterization of UCA, enabling us to prove several new state-of-the-art results.
- We consider practical restrictions that make combinatorial assignment problems tractable and/or approximable in polynomial time. This work includes e.g., *bounded combinatorial assignment*, in which bundles of size larger than some constant are given value zero.
- We explore combinatorial assignment generalizations with *externalities* (i.e., cross-bundle value dependencies) that we analyze and develop exponential-time algorithms for.
- We investigate the *balanced* (also called mixed) egalitarian and utilitarian combinatorial assignment problem.

2 RELATED WORK

We first turn to the *winner determination problem* (WDP) for combinatorial auctions—a canonical combinatorial assignment problem in which the goal is to maximize an auctioneer's profit. For the WDP, [24] provided several complexity bounds for various application-focused restrictions. [14] analyzed how the WDP's complexity is affected by using bidding languages, and [11] studied restrictions to hypergraphs describing the bids. [9] studied the WDP's complexity when the goal is instead to find an equilibrium.

For *coalition structure generation* (CSG) [26]—in which agents are partitioned into subsets called *coalitions*—[1, 2] considered different compactly representable problems. [31] analyzed CSG over graphs, and [17] investigated rule-based representations.

Related to our work on cross-bundle dependencies is CSG with externalities, which adds cross-coalitional effects to the coalitional values [23]. For this problem, only a few concise representations have so far been considered, such as the tree-based one by [32]. Our work on externalities can also be applied to such problems.

Another important combinatorial assignment problem is the *generalized assignment problem* (GAP) [10], which [5] surveyed. In the GAP, all items are given weights unique to the different alternatives, and the alternatives have capacities that denote the maximum total weight of items that can be assigned to them. The corresponding *value function* is *additive*, so there are no synergies, making the problem approximable [29] and more similar to knapsack problems.

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Table 1: Results for the k-bounded case. Here $l = \max(n, m)$.

	$k = 1$	$k \geq 2$	$m \geq 2$	m and k fixed
UCA	$O(l^3 m)$	NP-hard	NP-hard	$(3n)^{km} O(m)$
ECA	$O\big(m\sqrt{l^5}\log(l)\big)$	NP-hard	NP-hard	$(3n)^{km} O(km^2)$

Similar to our *k-bounded combinatorial assignment*—in which bundles of size larger than k are given value zero—are *k-additive domains* [7]. Specifically, k-additive problems are the UCA instances that can be represented by hypergraphs with hyperedges of size at most $k + 1$. Moreover, [28] proved several complexity results for *quadratic value functions*, which is equivalent to 2-bounded UCA.

For ECA and mixed welfare, [30] outlines corresponding egalitarian and balanced versions of CSG, and [6] presents maximizing egalitarian welfare as a way to offer a level of fairness in a system.

Finally, [18] studied a course assignment problem in which the goal is to find an equilibrium, and [3] provided complexity results for a related fair division resource allocation problem.

3 COMBINATORIAL ASSIGNMENT

In this paper we consider combinatorial assignment—i.e., the class of problems in which a set $N = \{1, \ldots, n\}$ of indivisible items (e.g., goods/agents) have to be distributed in bundles (i.e., partitioned) among a set $M = \{1, \ldots, m\}$ of alternatives (e.g., buyers/jobs) to maximize social welfare. The resulting ordered size-m partition of such a problem is called a *combinatorial assignment*.

Definition 3.1. *The tuple $(B_1, \ldots, B_m)$ is a* combinatorial assignment *over N if $B_i \cap B_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{m} B_i = N$.*

We use Π_N to denote the set of all combinatorial assignments over N, and define $\Pi_N^k = \{C \in \Pi_N : |C| = k\}$ for $k \in \{1, ..., m\}$. *The combinatorial assignment problem* can now be defined as follows.

The Combinatorial Assignment Problem

Input: A set of n items N, a set of m alternatives M, and a function (called the *social welfare function*) $\Phi : \Pi_N^m \to \mathbb{R}$.
Output: A *combinatorial assignment* $(B_1, \ldots, B_m)$ that maximizes the *value* (or *social welfare*) $\Phi((B_1, \ldots, B_m))$.

For UCA—the most central problem that we explore—the social welfare function is defined as

$$\Phi((B_1, \ldots, B_m)) = \sum_{i=1}^{m} v(B_i, i),$$

where v is a function $v : 2^N \times M \to \mathbb{R}$ (called the *value function*). Due to this welfare function, UCA is equivalent to *simultaneous coalition structure generation and assignment* [21] when the items are viewed as agents. It also generalizes other notable problems in a straightforward fashion, such as many of the aforementioned combinatorial assignment problems. UCA is NP-hard [20], and can be solved with *dynamic programming* (DP) in $O^*(3^n)$ [22] (the notation $O^*(\cdot)$ hides polynomial factors).

For ECA (which can also be solved with DP), with the same function v, the social welfare function is defined with

$$\Phi((B_1, \ldots, B_m)) = \min_{i=1}^{m} v(B_i, i).$$

4 RESULTS AND CONCLUSIONS

We analyzed restrictions and concise representations for several combinatorial assignment problems—the problems in which the goal is to partition n items into bundles among m alternatives to maximize social welfare. We considered both utilitarian (i.e., maximize the total output) and egalitarian (i.e., maximize the worst-case allocation) welfare functions. For various concise representations, generalizations and hypergraph characterizations of them, we provided novel algorithms and new complexity results.

In our hypergraph characterization, the value function is represented using a hypergraph called the *synergy hypergraph*, and we denoted the corresponding problem UCA*. For UCA*, the value of a bundle equals the sum of all of the (non-negative) synergies for every subset of the items in the bundle. We found that the case with two alternatives admits a polynomial-time algorithm by computing a cut of minimum cost in the hypergraph using an algorithm by [13]. However, more alternatives lead to intractability.

Theorem 4.1. *UCA* with $m = 2$ is solvable in polynomial time, while UCA* with $m \geq 3$ is NP-hard.*

Using this result, we were able to construct improved (faster than DP's $O^*(3^n)$) exp-time algorithms for some of the hard cases.

Theorem 4.2. *UCA* with $m = 3$, $m = 4$ and $m = 5$ is solvable in $O^*(1.89^n)$, $O^*(2^n)$ and $O^*(2.89^n)$ time, respectively.*

Under the aforementioned k-bounded restriction, we found that the problem remains NP-hard to approximate to some constant in polynomial time. On the positive side, k-bounded non-negative UCA can be approximated within $\frac{k+2}{2}$ of the optimum in polynomial time. Table 1 summarizes our other results for this representation.

Finally, we explored complexity bounds for generalizations with *balanced* utilitarian and egalitarian welfare functions (BCA), and *externalities* (XCA), for which a bundle's value can depend on other bundles' values. For BCA, we presented an exact $O^*(4^n)$ time algorithm. For a relaxation of XCA—denoted XCA*—we found that externalities makes the problem significantly harder, providing a lower bound under the *exponential time hypothesis* [12].

Theorem 4.3. *XCA* cannot be solved in $2^{o(n \log m)}$ time unless the exponential time hypothesis is false.*

While, in general, combinatorial assignment problems are computationally hard, we found that certain restrictions lead to tractability, allowing either polynomial-time exact and approximate algorithms, or non-trivial exponential-time algorithms. Thus, investigating structural restrictions is an interesting future direction. One possibility is to study synergy hypergraphs whose associated primal, dual, and incidence graphs have bounded treewitdh; even more general parameters like primal and dual hypertree-width are also interesting. A combination of parameters can also be considered, e.g., the cases with bounded bundle-size and bounded hypertree-width.

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