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Eulerian polynomials

Friedrich Hirzebruch

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Abstract. A short report on my lecture at the Conference in Honor of W. Killing in Münster on December 7, 2007, together with some remarks on the preceding paper by Arjeh M. Cohen.

I reported on Euler's famous paper *Remarques sur un beau rapport entre les series des puissances tant direct que reciproques*, Académie des sciences de Berlin, Lu en 1749, Opera Omnia Serie I, Bd. 15, pp. 70-90.

In modern terminology Euler introduces the alternating ζ-function

$$\varphi(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots.$$

This series converges for $\mathrm{Re}(s) > 0$. The function $\varphi(s)$ can be holomorphically extended to the whole s-plane. It is related to the ζ-function:

$$\varphi(s) = (1 - 2^{1-s})\zeta(s).$$

Euler wants to calculate $\varphi(-n)$ for $n = 0, 1, 2, 3, \dots$. For this purpose he introduces the Eulerian polynomials $P_n(t)$ by

$$\sum_{k=0}^{\infty} (k+1)^n t^k = \frac{P_n(t)}{(1-t)^{n+1}} \tag{1}$$

and writes down the polynomials up to $n = 6$:

$$\begin{aligned}
P_0(t) &= 1,\\
P_1(t) &= 1,\\
P_2(t) &= 1 + t,\\
P_3(t) &= 1 + 4t + t^2,\\
P_4(t) &= 1 + 11t + 11t^2 + t^3,\\
P_5(t) &= 1 + 26t + 66t^2 + 26t^3 + t^4,\\
P_6(t) &= 1 + 57t + 302t^2 + 302t^3 + 57t^4 + t^5.
\end{aligned}$$

Of course, the series (1) converges only for $|t| < 1$. But for Euler

(2) $$\varphi(-n) = P_n(-1)2^{-n-1}.$$

Using (1) one obtains easily the exponential generating function for the Eulerian polynomials

(3) $$\sum_{n=0}^{\infty} P_n(t)\frac{x^n}{n!} = \frac{(1-t)e^{(1-t)x}}{1-te^{(1-t)x}}$$

and thus

(4) $$\sum_{n=0}^{\infty} P_n(-1)\frac{x^n}{n!} = 1 + \mathrm{tgh}(x).$$

Therefore, according to Euler,

(5) $$\varphi(0) = \frac{1}{2}, \quad \varphi(-n) = \frac{\mathrm{tgh}^{(n)}(0)}{2^{n+1}} \text{ for } n > 0.$$

Euler compares $\varphi(-n)$ with $\varphi(n+1)$, well known to him for n odd. For $n > 0$ even, $\varphi(-n) = 0$. In this way, Euler conjectures Riemann's functional equation for the ζ-function.

We write

(6) $$P_n(t) = \sum_{k=0}^{n-1} W_{n,k}t^k.$$

By multiplying (1) with t and differentiating we obtain

(7a) $$P_{n+1}(t) = P_n(t)(1+nt) + t(1-t)P_n'(t)$$

(7b) $$W_{n+1,k}(t) = (k+1)W_{n,k} + (n+k+1)W_{n,k-1}$$

and prove by induction

$$W_{n,k} = \text{number of permutations of } \{1, 2, \ldots, n\} \text{ with } k \text{ ascents.}$$

The Eulerian numbers $W_{n,k}$ occur in this form in the combinatorial literature (modified Pascal triangle (7b)) usually without mention of Euler's paper. We have

(8) $$P_n(t) = P(A_{n-1}, t)$$

in the notation of the preceding paper. The first formula in Theorem 4 of the preceding paper follows from (1) and the binomial development

$$(k+1)^{m+1} - (k+1-1)^{m+1} = \sum_{i=0}^{m}(k+1)^{m-i}(-1)^i\binom{m+1}{i+1}.$$

Euler says in his paper that he can deal in the same way with the function

$$L(s) = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-s}.$$

He does not carry out details. By a similar induction as above we have

$$\sum_{k=0}^{\infty}(2k+1)^n t^k = \frac{P(B_n,t)}{(1-t)^{n+1}}.$$

The second formula in Theorem 4 of the preceding paper follows from the binomial development

$$(2(k+1)-1)^n = \sum_{i=0}^{n} 2^{n-i}(k+1)^{n-i}(-1)^i\binom{n}{i}.$$

The exponential generating function for the polynomials $P(B_n,t)$ is given by the formula

$$\sum_{n=0}^{\infty} P(B_n,t)\frac{x^n}{n!} = \frac{(1-t)e^{(1-t)x}}{1-te^{2(1-t)x}}$$

and thus

$$\sum_{n=0}^{\infty} P(B_n,-1)\frac{x^n}{n!} = \frac{1}{\cosh(2x)}$$

and, according to Euler,

$$L(-n) = \frac{P(B_n,-1)}{2^{n+1}} = \frac{E_n}{2},$$

where the Euler number E_n is the n-th derivative of $\frac{1}{\cosh(x)}$ at $x=0$. Again, Euler conjectures the functional equation for $L(s)$.

Some algebraic geometry

1) Consider the projective algebraic variety $P_1^n = P_1 \times \cdots \times P_1$ where P_1 is the complex projective line. On the i-th factor we use homogeneous coordinates (x_i, y_i) and study the divisor D given by $x_1 x_2 \cdots x_n = 0$. Then the dimension of the space of meromorphic functions on P_1^n whose divisor plus kD ($k \geqq 0$) is non-negative equals $(k+1)^n$. This is the Hilbert polynomial for D and so $\sum_{k=0}^{\infty}(k+1)^n t^k$ is the Hilbert series of commutative algebra for the divisor D. For the divisor $2D$ we obtain the Hilbert series $\sum_{k=0}^{\infty}(2k+1)^n t^k$. In this way the polynomials $P(A_{n-1},t)$ and $P(B_n,t)$ are objects of algebraic geometry.

2) The toric varieties associated to the polytopes coming from the Coxeter systems studied in the preceding paper have only even-dimensional cohomology. The coefficients of the Eulerian polynomials are the Betti numbers. These varieties have only Hodge numbers $h^{p,p}$. All other $h^{p,q}$ vanish. Therefore $h^{p,p}$ is the $(2p)$-th Betti number, the p-th coefficient of the Eulerian polynomial. In general, the signature equals $\sum_{p,q}(-1)^q h^{p,q}$. Thus in our case the signature of the toric varieties equals the value of the Eulerian polynomial at -1, exactly the value Euler had to use.

In the case of A_{n-1} the signature is the n-th derivative of $\operatorname{tgh}(x)$ at $x=0$, for B_n it is $2^n E_n$. There is a vast literature on these polytopes and the corresponding toric varieties. The toric variety belonging to A_{n-1} occurs in a paper by A. Losev and Y. Manin [3] as a moduli space of pointed curves.

3) In my book [1] I introduce the χ_y-genus for a projective algebraic manifold X of dimension d as

$$\chi_y(X) = \sum_{p=0}^{d} \chi^p(X) y^p$$

where

$$\chi^p(X) = \sum_{q=0}^{d} (-1)^q h^{p,q}.$$

One of the main results of the book is that $\chi_y(X)$ is the genus associated with the characteristic power series $x/f(x)$, where

$$f(x) = -\frac{e^{-(1+y)x} - 1}{1 + ye^{-(1+y)x}}.$$

The power series $x/f(x)$ was denoted in [1], section 10.2, by $Q(y;x)$. If we replace y by $-t$ and x by $-x$, then $x/f(x)$ goes over in $x/\tilde{f}(x)$ with

$$\begin{aligned} \tilde{f}(x) &= \frac{e^{(1-t)x} - 1}{1 - te^{(1-t)x}} \\ &= \sum_{n=1}^{\infty} P_n(t) \frac{x^n}{n!} \end{aligned}$$

(compare (3) where the summation runs from 0 to ∞). Thus $x/\tilde{f}(x)$ is the characteristic power series for the genus $(-1)^d \chi_{-t}(X)$ with $d = \dim(X)$. Therefore the Eulerian polynomial $P_n(t)$ equals $(-1)^{n-1}\chi_{-t}(X)$ for a variety X of dimension $n-1$ with total Chern class $(1+g)^{-1}$ and $g^{n-1}[X] = n!$ ($g \in H^2(X, \mathbf{Z})$). The Θ-divisor in a principally polarized abelian variety of dimension n satisfies these conditions: generically the Θ-divisor is smooth and represents a cohomology class of dimension 2 which restricted to the Θ-divisor plays the role of g. The fact that the Eulerian polynomials $P_n(t)$ are essentially the χ_y-genera of the Θ-divisors was mentioned in [2].

4) A general reference for the following remarks is [1]. Let D be a divisor on a projective algebraic manifold X of dimension n. We denote the characteristic class of D by $g \in H^2(X, \mathbf{Z})$. For the divisor kD with $k \in \mathbf{Z}$ we consider the sheaf of local meromorphic functions f (with divisor (f)) such that

$$(f) + kD \geqq 0$$

and denote it by $\mathcal{O}(kD)$. The holomorphic Euler number $\chi(X, \mathcal{O}(kD))$ depends only on k and g. We write

$$\chi(X, \mathcal{O}(kD)) = \chi(X, kg).$$

This is a polynomial in k of degree at most n, namely

$$\chi(X, kg) = \frac{g^n}{n!}[X]k^n + \cdots + \chi(X, \mathcal{O})$$

where $\mathcal{O}$ is the structure sheaf of X.

For the Hilbert series

$$\sum_{k=0}^{\infty} \chi(X, kg) t^k \tag{9}$$

we introduce the polynomial $H(t)$ of degree $\leqq n$ defined by

$$\sum_{k=0}^{\infty} \chi(X, kg) t^k = \frac{H(t)}{(1-t)^{n+1}} \tag{10}$$

called H-vector in commutative algebra. Now suppose that g is ample and the first Chern class of X equals

$$c_1 = \lambda g \text{ with } \lambda > 0 \text{ and integral.}$$

Then by Kodaira's vanishing theorem (see [1]) the cohomology groups $H^i(X, \mathcal{O}(kD))$ vanish for $i > 0$ and $k > -\lambda$. Therefore

$$\begin{aligned} &\chi(X, kg) = \dim H^0(X, \mathcal{O}(k)) \text{ for } k > -\lambda \\ &\chi(X, kg) = 0 \text{ for } -\lambda < k < 0 \\ &\chi(X, 0 \cdot g) = 1 \end{aligned} \tag{11}$$

We also need the following consequence of Serre's duality

$$\chi(X, -kg) = (-1)^n \chi(X, (k-\lambda)g). \tag{12}$$

The following theorem is probably well known. We shall prove it by the R-R-theorem of [1]. Compare Lemma 2(v) of the preceding paper concerning the polynomials $P(A_{n-1}, t)$ and $P(B_n, t)$.

Theorem. *Under the above assumptions (g ample, λ positive) the polynomial $H(t)$ has degree $n+1-\lambda$ and satisfies*

$$t^{n+1-\lambda} H(\frac{1}{t}) = H(t).$$

Proof. By R-R the Hilbert series (9) equals

$$\left(\sum_{k=0}^{\infty} e^{kg} t^k \right) T = \frac{1}{1-te^g} T$$

where T is the total Todd class. This expression has to be evaluated on the fundamental cycle of X. In the same way $\sum_{k=-1}^{-\infty} \chi(X, kg) t^k$ is given by

$$\frac{t^{-1}e^{-g}}{1-t^{-1}e^{-g}} T = \frac{-1}{1-te^g} T.$$

Thus

$$\sum_{k=0}^{\infty} \chi(X, kg) t^k = - \sum_{k=-1}^{-\infty} \chi(X, kg) t^k.$$

By changing the summation index from k to $-k$ and using (12) and (11)

$$\begin{aligned}\sum_{k=0}^{\infty}\chi(X,kg)t^k &= (-1)^{n+1}\sum_{k=1}^{\infty}\chi(X,(k-\lambda)g)t^{-k}\\ &= (-1)^{n+1}\sum_{k=\lambda}^{\infty}\chi(X,(k-\lambda)g)t^{-k}\\ &= (-1)^{n+1}\sum_{k=0}^{\infty}\chi(X,kg)t^{-k-\lambda}.\end{aligned}$$

By the definition of $H(t)$, see (10), this gives

$$\begin{aligned}H(t)(1-t)^{-n-1} &= (-1)^{n+1}t^{-\lambda}H(t^{-1})(1-t^{-1})^{-n-1}\\ &= t^{n+1-\lambda}H(\frac{1}{t})(1-t)^{-n-1}\end{aligned}$$

which concludes the proof. □

In the two examples in 1) the manifold X equals P_1^n, the first Chern class is $2g$ where g is the characteristic class of D. In the first example D is our divisor for the Hilbert series, in the second example it is $2D$. Therefore $\lambda = 2$ and $\lambda = 1$, respectively. The polynomials $H(t)$ have degrees $n-1$ and n, in accordance with the theorem. For the n-dimensional complex projective space X and the ample generator g of $H^2(X, \mathbf{Z})$ we have $\lambda = n+1$ and $H(t) = 1$.

About the material of this paper I had many fruitful discussions with V. Gorbounov.

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Friedrich Hirzebruch
Max-Planck-Institut für Mathematik
Vivatsgasse 7, 53111 Bonn, Germany
E-mail: `hirzebruch@mpim-bonn.mpg.de`