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On k-Fibonacci Brousseau Sums

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Abstract

Using elementary methods, we provide formulas for evaluating the Brousseau sum $\sum_{i=1}^{n} i^p F_{k,i}$ and the shifted Brousseau sum $\sum_{i=1}^{n} i^p F_{k,m+i}$ for all integers $m, p \geq 0$, where $(F_{k,i})_{i\geq 0}$ is the k-Fibonacci sequence defined by the two-term linear recurrence $F_{k,i} = kF_{k,i-1} + F_{k,i-2}$ for $i \geq 2$ with initial values $F_{k,0} = 0$ and $F_{k,1} = 1$.

1 Introduction

In the second issue of *Fibonacci Quarterly* in 1963, Brousseau [4] proposed a problem to discover an expression for the Fibonacci sum of the form

$$\sum_{i=1}^{n} i^3 F_i,$$

where F_i is the i^{th} Fibonacci number. In the following year, Erbacher and Fuchs [7] gave a solution for this problem in terms of F_{n+2} and F_{n+3}. Later, Ledin [12], Brousseau [5], and Zeitlin [16] developed various methods to determine expressions for the Brousseau sum

$$\sum_{i=1}^{n} i^p F_i,$$

where p is a non-negative integer. Ledin showed that the solution of Erbacher and Fuchs can be expressed in the form

$$\sum_{i=1}^{n} i^3 F_i = (n^3 - 6n^2 + 24n - 50)F_{n+1} + (n^3 - 3n^2 + 15n - 31)F_n + 50. \tag{1}$$

Recently, Ollerton [13], Shannon [14], Hendel [11], and Adegoke [1] derived expressions for such sums. Dresden [6] used a different technique to find the sum $\sum_{i=1}^{n} i^p F_i$ using just the binomial coefficients. Motivated by this, we develop formulas for the sums

$$\sum_{i=1}^{n} i^p F_{k,i}, \text{ and } \sum_{i=1}^{n} i^p F_{k,m+i},$$

for integers $m, p \geq 0$, where $F_{k,i}$ is the i^{th} k-Fibonacci number. For a positive integer k, the k-Fibonacci sequence (see [8]) $(F_{k,n})_{n\geq 0}$ is defined by the two-term linear recurrence

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \tag{2}$$

with initial values $F_{k,0} = 0$ and $F_{k,1} = 1$. The numbers $F_{k,n}$ are sometimes called the "metallic" or "metallonacci" numbers. The numbers $F_{1,n}$ are the "regular" Fibonacci numbers and $F_{2,n}$ are the Pell numbers P_n. The Pell numbers $F_{2,n}$ are sometimes called the *silver* Fibonacci numbers, and the numbers $F_{3,n}$ are sometimes called the *bronze* Fibonacci numbers. For $k = 1, 2, 3$, and 4, the numbers $F_{k,n}$ are A000045, A000129, A006190, and A001076, respectively, in the OEIS [15]. The k-Fibonacci numbers can be extended to negative subscripts by

$$F_{k,n} = F_{k,n+2} - kF_{k,n+1}, \text{ for } n < 0.$$

The Lucas sequence $\left(L_n\right)_{n\geq 0}$ and the Pell-Lucas sequence $\left(Q_n\right)_{n\geq 0}$ are defined, respectively, by

$$L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2,$$

and

$$Q_0 = Q_1 = 2,\ Q_n = 2Q_{n-1} + Q_{n-2} \text{ for } n \geq 2.$$

The numbers L_n and Q_n are A000032 and A002203, respectively, in the OEIS. Falcón [9] proved that the 4-Fibonacci numbers $F_{4,n}$ are just F_{3n}/F_3, the 11-Fibonacci numbers $F_{11,n}$ are just F_{5n}/F_5, the 29-Fibonacci numbers $F_{29,n}$ are just F_{7n}/F_7, and so on. In general, for all odd indexed Lucas numbers L_m, the L_m-Fibonacci numbers $F_{L_m,n}$ are just F_{mn}/F_m. The sequence $1, 4, 11, 29, 76, \ldots$ is A002878 in the OEIS. Falcón [9] also proved that the 14-Fibonacci numbers $F_{14,n}$ are just P_{3n}/P_3, the 82-Fibonacci numbers $F_{82,n}$ are just P_{5n}/P_5, the 478-Fibonacci numbers $F_{478,n}$ are just P_{7n}/P_7, and so on. In general, for all odd indexed Pell-Lucas numbers Q_m, the Q_m-Fibonacci numbers are just P_{mn}/P_m. The sequence $2, 14, 82, 478, 2786, \ldots$ is A077444 in the OEIS.

Hendel proved that

$$2 \cdot \sum_{i=1}^{n} i^3 P_i = (n^3 - 3n^2 + 6n - 7)P_{n+1} + (n^3 + 3n - 3)P_n + 7. \tag{3}$$

We can see the clear similarities between Eqs. (1) and (3). For further clarity, we rewrite Eqs. (1) and (3) as

$$\sum_{i=1}^{n} i^3 F_i = (n^3 - 6n^2 + 24n - 50)F_{n+1} + ((n+1)^3 - 6(n+1)^2 + 24(n+1) - 50)F_n + 50, \tag{4}$$

and

$$2 \cdot \sum_{i=1}^{n} i^3 P_i = (n^3 - 3n^2 + 6n - 7)P_{n+1} + ((n+1)^3 - 3(n+1)^2 + 6(n+1) - 7)F_n + 7. \tag{5}$$

The identity (4) appears in the OEIS at A259546. If we use the 3-Fibonacci numbers as another example, then we would have

$$3\cdot\sum_{i=1}^{n} i^3 F_{3,i} = \left(n^3 - 2n^2 + \frac{8}{3}n - \frac{22}{9}\right)F_{3,n+1} + \left((n+1)^3 - 2(n+1)^2 + \frac{8}{3}(n+1) - \frac{22}{9}\right)F_{3,n} + \frac{22}{9}. \tag{6}$$

Eqs. (4), (5), and (6) naturally suggest the following interesting generalization about the Brousseau sums of the k-Fibonacci numbers:

$$k \cdot \sum_{i=1}^{n} i^p F_{k,i} = (C_k^{(p)}(n))F_{k,n+1} + (C_k^{(p)}(n+1))F_{k,n} - C_k^{(p)}(0), \tag{7}$$

where $C_k^{(p)}(n)$ is a "coefficient polynomial" in n of degree p with rational coefficients. The main task is to find an expression for the polynomial $C_k^{(p)}(n)$. We do this using some simple recursion formulas involving just binomial coefficients. We use a similar technique as that of Dresden. Throughout this paper, we assume $\binom{0}{0} = 1$.

2 k-Fibonacci numbers and powers

We begin with the following set of identities, which are similar to those with the "regular" Fibonacci numbers [6].

$$
\begin{aligned}
F_{k,n} &= n + \sum_{i=1}^{n} \left(ki - 2 \cdot \binom{1}{1} \right) F_{k,n-i},\\
F_{k,n} &= n^2 + \sum_{i=1}^{n} \left(ki^2 - 2 \cdot \binom{2}{1} i \right) F_{k,n-i},\\
F_{k,n} &= n^3 + \sum_{i=1}^{n} \left(ki^3 - 2 \cdot \binom{3}{1} i^2 - 2 \cdot \binom{3}{3} \right) F_{k,n-i},\\
F_{k,n} &= n^4 + \sum_{i=1}^{n} \left(ki^4 - 2 \cdot \binom{4}{1} i^3 - 2 \cdot \binom{4}{3} i \right) F_{k,n-i},\\
F_{k,n} &= n^5 + \sum_{i=1}^{n} \left(ki^5 - 2 \cdot \binom{5}{1} i^4 - 2 \cdot \binom{5}{3} i^2 - 2 \cdot \binom{5}{5} \right) F_{k,n-i},
\end{aligned}
$$

and so on. Our first theorem involves the generalization of these formulas.

Theorem 1. *For all integers $n, p \geq 1$, we have*

$$
F_{k,n} = n^p + \sum_{i=1}^{n} \left(ki^p - 2 \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} i^{p-2j+1} \right) F_{k,n-i}. \tag{8}
$$

Proof. We use induction on n. When $n = 1$, the left-hand side of Eq. (8) is $F_{k,1} = 1$ and the right-hand side is

$$
\begin{aligned}
&1 + \sum_{i=1}^{1} \left(ki^p - 2 \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} i^{p-2j+1} \right) F_{k,1-i}\\
&= 1 + \left(k - 2 \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} \right) F_{k,0}\\
&= 1 + 0\\
&= 1.
\end{aligned}
$$

When $n = 2$, the left-hand side of Eq. (8) is $F_{k,2} = k$ and the right-hand side is

$$\begin{aligned}
&2^p + \sum_{i=1}^{2}\left(ki^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil}\binom{p}{2j-1} i^{p-2j+1}\right)F_{k,2-i} \\
&= 2^p + \left(k - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil}\binom{p}{2j-1}\right)F_{k,1} + \left(k2^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil}\binom{p}{2j-1}2^{p-2j+1}\right)F_{k,0} \\
&= 2^p + (k - 2\cdot 2^{p-1}) + 0 \\
&= k.
\end{aligned}$$

Thus, Eq. (8) holds for $n = 1$ and $n = 2$. Now fix $n \geq 2$. Assume that Eq. (8) holds for $n-1$ and n. Then

$$F_{k,n} = n^p + \sum_{i=1}^{n}\left(ki^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil}\binom{p}{2j-1} i^{p-2j+1}\right)F_{k,n-i}, \tag{9}$$

and

$$F_{k,n-1} = (n-1)^p + \sum_{i=1}^{n-1}\left(ki^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil}\binom{p}{2j-1} i^{p-2j+1}\right)F_{k,n-1-i}. \tag{10}$$

Since $F_{k,n-i} = 0$ for $i = n$, Eq. (9) can be put in the form

$$F_{k,n} = n^p + \sum_{i=1}^{n-1}\left(ki^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil}\binom{p}{2j-1} i^{p-2j+1}\right)F_{k,n-i}. \tag{11}$$

Multiplying Eq. (11) by k and adding it to Eq. (10), we get

$$kF_{k,n} + F_{k,n-1} = kn^p + (n-1)^p + \sum_{i=1}^{n-1}\left(ki^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil}\binom{p}{2j-1} i^{p-2j+1}\right)(kF_{k,n-i} + F_{k,n-i-1}).$$

Using the k-Fibonacci recurrence (2), this becomes

$$F_{k,n+1} = kn^p + (n-1)^p + \sum_{i=1}^{n-1}\left(ki^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil}\binom{p}{2j-1} i^{p-2j+1}\right)F_{k,n-i+1}. \tag{12}$$

Using the binomial expansion, we have the identity

$$(n+1)^p - (n-1)^p = 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil}\binom{p}{2j-1} n^{p-2j+1},$$

and hence

$$(n-1)^p = (n+1)^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} n^{p-2j+1}. \tag{13}$$

Now using Eqs. (12) and (13), we obtain

$$\begin{aligned}
F_{k,n+1} &= kn^p + (n+1)^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} n^{p-2j+1} \\
&\quad + \sum_{i=1}^{n-1} \left(ki^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} i^{p-2j+1} \right) F_{k,n+1-i} \\
&= (n+1)^p + \left(kn^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} n^{p-2j+1} \right) \\
&\quad + \sum_{i=1}^{n-1} \left(ki^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} i^{p-2j+1} \right) F_{k,n+1-i}.
\end{aligned}$$

Since $F_{k,1} = 1$, we rewrite this as

$$F_{k,n+1} = (n+1)^p + \sum_{i=1}^{n} \left(ki^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} i^{p-2j+1} \right) F_{k,n+1-i}.$$

Since $F_{k,n+1-i} = 0$ when $i = n+1$, we may simply add the corresponding term to the summand on the right-hand side of the above equation to get

$$F_{k,n+1} = (n+1)^p + \sum_{i=1}^{n+1} \left(ki^p - 2\sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} i^{p-2j+1} \right) F_{k,n+1-i},$$

and this completes the induction step. This concludes the proof. □

3 Convolutions

Applying Theorem 1 along with the k-Fibonacci sum [8, Proposition 8]

$$\sum_{i=0}^{n} F_{k,n} = \frac{1}{k}\big(F_{k,n+1} + F_{k,n} - 1\big),$$

we can recursively find the following convolution identities:

$$
\begin{aligned}
k \cdot \sum_{i=0}^{n} F_{k,n-i} &= F_{k,n+1} + F_{k,n} - 1, \\
k \cdot \sum_{i=0}^{n} iF_{k,n-i} &= \frac{1}{k}\big(2F_{k,n+1} + (-k+2)F_{k,n} - (kn+2)\big) \\
&= \left(\frac{2}{k}\right)F_{k,n+1} + \left(\frac{-k+2}{k}\right)F_{k,n} - \frac{kn+2}{k}, \\
k \cdot \sum_{i=0}^{n} i^2F_{k,n-i} &= \frac{1}{k^2}\big(8F_{k,n+1} + (k^2-4k+8)F_{k,n} - (k^2n^2+4kn+8)\big) \\
&= \left(\frac{8}{k^2}\right)F_{k,n+1} + \left(\frac{k^2-4k+8}{k^2}\right)F_{k,n} - \frac{k^2n^2+4kn+8}{k^2}, \\
k \cdot \sum_{i=0}^{n} i^3F_{k,n-i} &= \frac{1}{k^3}\big((2k^2+48)F_{k,n+1} + (-k^3+8k^2-24k+48)F_{k,n} \\
&\quad - (k^3n^3+6k^2n^2+24kn+2k^2+48)\big), \\
&= \left(\frac{2k^2+48}{k^3}\right)F_{k,n+1} + \left(\frac{-k^3+8k^2-24k+48}{k^3}\right)F_{k,n} \\
&\quad - \frac{k^3n^3+6k^2n^2+24kn+2k^2+48}{k^3},
\end{aligned}
$$

and so on. Each sum on the left-hand side of the above set of equations is a convolution of the powers of i and the k-Fibonacci numbers. We define

$$
T_{k,n}^{(p)} = \begin{cases} \sum_{i=0}^{n} F_{k,n-i}, & \text{if } p = 0; \\ \sum_{i=0}^{n} i^p F_{k,n-i}, & \text{if } p \geq 1. \end{cases} \tag{14}
$$

A pattern is evident in the above set of equations. Note that each equation is of the form

$$
k \cdot T_{k,n}^{(p)} = \Phi_k^{(p)}(0)F_{k,n+1} + \Phi_k^{(p)}(-1)F_{k,n} - \Phi_k^{(p)}(n), \tag{15}
$$

where $\Phi_k^{(p)}(n)$ is a polynomial in n of degree p. To find an explicit formula for $\Phi_k^{(p)}(n)$, we must define the sequence $\big(A_k^{(p)}\big)_{p\geq 0}$ as follows:

Definition 2. The sequence $\big(A_k^{(p)}\big)_{p\geq 0}$ of numbers is defined by the recurrence

$$
A_k^{(p)} = \begin{cases} 1, & \text{if } p = 0; \\ \frac{2}{k}\sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} A_k^{(p-2j+1)}, & \text{if } p \geq 1. \end{cases} \tag{16}
$$

The recurrence $A_k^{(p)}$ generates the sequence

$$1, \frac{2}{k}, \frac{8}{k^2}, \frac{2k^2+48}{k^3}, \frac{32k^2+384}{k^4}, \ldots.$$

Note that these numbers are the coefficients of $F_{k,n+1}$ in the above set of equations. The numbers $A_k^{(p)}$ for $p = 1, 2, 3$, and 4 are given below:

(i) $A_1^{(p)} : 1, 2, 8, 50, 416, 4322, 53888, \ldots$. This sequence is A000557 in the OEIS [15].

(ii) $A_2^{(p)} : 1, 1, 2, 7, 32, 181, 1232, \ldots$. This sequence is A006154 in the OEIS [15].

(iii) $A_3^{(p)} : 1, \frac{2}{3}, \frac{8}{9}, \frac{22}{9}, \frac{224}{27}, \frac{2774}{81}, \frac{13952}{81}, \ldots$.

(iv) $A_4^{(p)} : 1, \frac{1}{2}, \frac{1}{2}, \frac{5}{4}, \frac{7}{2}, \frac{47}{4}, \frac{197}{4}, \ldots$.

From the last equation in the set of convolution identities given above, we recognize that

$$\begin{aligned}\Phi_k^{(3)}(n) &= n^3 + \frac{6}{k}n^2 + \frac{24}{k^2}n + \frac{2k^2+48}{k^3} \\ &= 1 \cdot \binom{3}{0}n^3 + \frac{2}{k} \cdot \binom{3}{1}n^2 + \frac{8}{k^2} \cdot \binom{3}{2}n + \frac{2k^2+48}{k^3} \cdot \binom{3}{3},\end{aligned}$$

where $1, 2/k, 8/k^2, (2k^2+48)/k^3$ are the first four terms of the sequence $\left(A_k^{(p)}\right)_{p\geq 0}$. With all this in mind, we make the following definition:

Definition 3. For all integers $p \geq 0$, we define the polynomial $\Phi_k^{(p)}(n)$ as

$$\Phi_k^{(p)}(n) = \begin{cases} A_k^{(p)}, & \text{if } n = 0; \\ \sum_{r=0}^{p} A_k^{(r)} \binom{p}{r} n^{p-r}, & \text{if } n \neq 0. \end{cases} \tag{17}$$

Rewriting Eq. (15) in terms of $A_k^{(p)}$, we have the following theorem:

Theorem 4. *If $T_{k,n}^{(p)}$ is as defined in Eq. (14), then for all integers $p \geq 0$, we have*

$$k \cdot T_{k,n}^{(p)} = A_k^{(p)} F_{k,n+1} + \sum_{r=0}^{p} A_k^{(r)} \binom{p}{r} \left((-1)^{p-r} F_{k,n} - n^{p-r}\right). \tag{18}$$

Proof. We use induction on p. When $p = 0$, the left-hand side of Eq. (18) is

$$k \cdot T_{k,n}^{(0)} = k \sum_{i=0}^{n} F_{k,n-i} = F_{k,n+1} + F_{k,n} - 1,$$

and the right-hand side is

$$A_k^{(0)} F_{k,n+1} + A_k^{(0)}(F_{k,n} - 1) = \frac{1}{k}(F_{k,n+1} + F_{k,n} - 1).$$

Thus, Eq. (18) holds for $p = 0$. Now, fix $p \geq 1$. Assume that Eq. (18) holds for all non-negative integers less than p. First, we rewrite Eq. (8) in Theorem 1 as

$$F_{k,n} = n^p + k \sum_{i=1}^{n} i^p F_{k,n-i} - 2 \sum_{i=1}^{n} \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} i^{p-2j+1} F_{k,n-i}.$$

By switching the order of summation in the double summand, we get

$$F_{k,n} = n^p + k \sum_{i=1}^{n} i^p F_{k,n-i} - 2 \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} \sum_{i=1}^{n} i^{p-2j+1} F_{k,n-i}.$$

Since $i^p F_{k,n-i} = 0$ for $i = 0$, we can start the first summand at $i = 0$ instead of at $i = 1$, and thus we obtain

$$F_{k,n} = n^p + k \sum_{i=0}^{n} i^p F_{k,n-i} - 2 \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} \sum_{i=1}^{n} i^{p-2j+1} F_{k,n-i}. \tag{19}$$

If p is even, then Eq. (19) can be put in the form

$$F_{k,n} = n^p + k \sum_{i=0}^{n} i^p F_{k,n-i} - 2 \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} \sum_{i=0}^{n} i^{p-2j+1} F_{k,n-i}, \tag{20}$$

because the term corresponding to $i = 0$ in the last summand is $0^{p-2j+1} F_{k,n} = 0$ for all $j = 1, 2, \ldots, p/2$. On the other hand, if p is odd, then Eq. (19) can be written as

$$F_{k,n} = n^p + k \sum_{i=0}^{n} i^p F_{k,n-i} - 2 \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} \sum_{i=0}^{n} i^{p-2j+1} F_{k,n-i} + 2F_{k,n},$$

because the term corresponding to $i = 0$ in the last summation is $0^{p-2j+1} F_{k,n} = 0$ for $j \neq (p+1)/2$ and $F_{k,n}$ for $j = (p+1)/2$. Therefore, if p is odd, we have

$$-F_{k,n} = n^p + k \sum_{i=0}^{n} i^p F_{k,n-i} - 2 \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} \sum_{i=0}^{n} i^{p-2j+1} F_{k,n-i}. \tag{21}$$

Thus, from Eqs. (20) and (21), we conclude that

$$(-1)^p F_{k,n} = n^p + k \cdot T_{k,n}^{(p)} - 2 \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} T_{k,n}^{(p-2j+1)},$$

and hence

$$k \cdot T_{k,n}^{(p)} = (-1)^p F_{k,n} - n^p + \frac{2}{k} \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} k \cdot T_{k,n}^{(p-2j+1)}. \tag{22}$$

Now, by induction hypothesis, for $1 \le j \le \lceil \frac{p}{2} \rceil$, we have

$$k \cdot T_{k,n}^{(p-2j+1)} = A_k^{(p-2j+1)} F_{k,n+1} + \sum_{r=0}^{p-2j+1} A_k^{(r)} \binom{p-2j+1}{r} \big((-1)^{p-2j+1-r} F_{k,n} - n^{p-2j+1-r}\big).$$

Substituting this in Eq. (22), we get

$$\begin{aligned} k \cdot T_{k,n}^{(p)} = {} & (-1)^p F_{k,n} - n^p + \frac{2}{k} \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} A_k^{(p-2j+1)} F_{k,n+1} \\ & + \frac{2}{k} \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \binom{p}{2j-1} \sum_{r=0}^{p-2j+1} A_k^{(r)} \binom{p-2j+1}{r} \big((-1)^{p-2j+1-r} F_{k,n} - n^{p-2j+1-r}\big). \end{aligned}$$

Using the Definition 2, this can be written as

$$\begin{aligned} k \cdot T_{k,n}^{(p)} = {} & (-1)^p F_{k,n} - n^p + A_k^{(p)} F_{k,n+1} \\ & + \frac{2}{k} \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \sum_{r=0}^{p-2j+1} A_k^{(r)} \binom{p}{2j-1} \binom{p-2j+1}{r} \big((-1)^{p-2j+1-r} F_{k,n} - n^{p-2j+1-r}\big). \end{aligned}$$

If we execute the change of variable $r' = 2j + r - 1$, this becomes

$$\begin{aligned} k \cdot T_{k,n}^{(p)} = {} & (-1)^p F_{k,n} - n^p + A_k^{(p)} F_{k,n+1} \\ & + \frac{2}{k} \sum_{j=1}^{\lceil \frac{p}{2} \rceil} \sum_{r'=2j-1}^{p} A_k^{(r'-2j+1)} \binom{p}{2j-1} \binom{p-2j+1}{r'-2j+1} \big((-1)^{p-r'} F_{k,n} - n^{p-r'}\big). \end{aligned}$$

Now, by switching the order of summation, we obtain

$$\begin{aligned} k \cdot T_{k,n}^{(p)} = {} & (-1)^p F_{k,n} - n^p + A_k^{(p)} F_{k,n+1} \\ & + \frac{2}{k} \sum_{r'=1}^{p} \sum_{j=1}^{\lceil \frac{r'}{2} \rceil} A_k^{(r'-2j+1)} \binom{p}{2j-1} \binom{p-2j+1}{r'-2j+1} \big((-1)^{p-r'} F_{k,n} - n^{p-r'}\big). \end{aligned}$$

Using the well-known binomial identity $\binom{n}{r} = \binom{n}{n-r}$, we rewrite this as

$$\begin{aligned} k \cdot T_{k,n}^{(p)} = {} & (-1)^p F_{k,n} - n^p + A_k^{(p)} F_{k,n+1} \\ & + \frac{2}{k} \sum_{r'=1}^{p} \sum_{j=1}^{\lceil \frac{r'}{2} \rceil} A_k^{(r'-2j+1)} \binom{p}{p-2j+1} \binom{p-2j+1}{p-r'} \big((-1)^{p-r'} F_{k,n} - n^{p-r'}\big). \end{aligned} \tag{23}$$

Next, using the binomial identity [2, Identity 134, p. 67] $\binom{p}{q}\binom{q}{r} = \binom{p}{r}\binom{p-r}{p-q}$, we have

$$\binom{p}{p-2j-1}\binom{p-2j+1}{p-r'} = \binom{p}{p-r'}\binom{r'}{2j-1} = \binom{p}{r'}\binom{r'}{2j-1}.$$

When we substitute this in Eq. (23), we obtain

$$\begin{aligned}
k \cdot T_{k,n}^{(p)} &= (-1)^p F_{k,n} - n^p + A_k^{(p)} F_{k,n+1} \\
&\quad + \frac{2}{k} \sum_{r'=1}^{p} \sum_{j=1}^{\lceil \frac{r'}{2} \rceil} A_k^{(r'-2j+1)} \binom{p}{r'} \binom{r'}{2j-1} \big((-1)^{p-r'} F_{n,k} - n^{p-r'}\big) \\
&= (-1)^p F_{k,n} - n^p + A_k^{(p)} F_{k,n+1} \\
&\quad + \sum_{r'=1}^{p} \left(\frac{2}{k} \sum_{j=1}^{\lceil \frac{r'}{2} \rceil} \binom{r'}{2j-1} A_k^{(r'-2j+1)} \right) \binom{p}{r'} \big((-1)^{p-r'} F_{n,k} - n^{p-r'}\big).
\end{aligned}$$

Now, using Definition 2, this becomes

$$k \cdot T_{k,n}^{(p)} = A_k^{(p)} F_{k,n+1} + \big((-1)^p F_{k,n} - n^p\big) + \sum_{r'=1}^{p} A_k^{(r')} \binom{p}{r'} \big((-1)^{p-r'} F_{n,k} - n^{p-r'}\big)\big).$$

Since $A_k^{(r')} = 1$ at $r' = 0$, we conclude that

$$k \cdot T_{k,n}^{(p)} = A_k^{(p)} F_{k,n+1} + \sum_{r'=0}^{p} A_k^{(r')} \binom{p}{r'} \big((-1)^{p-r'} F_{n,k} - n^{p-r'}\big).$$

Hence, by induction, Eq. (18) holds for all $p \geq 0$. This completes the proof. □

4 The Brousseau sums

Let us begin this section with finding the identities about the Brousseau sums $\sum_{i=0}^{n} i^p F_{k,i}$ for $p = 1, 2, 3, \ldots$. Consider the case $p = 1$.

$$\begin{aligned}
k \cdot \sum_{i=0}^{n} i F_{k,i} &= k \cdot \sum_{i=0}^{n} (n-i) F_{k,n-i} \\
&= nk \cdot \sum_{i=0}^{n} F_{k,n-i} - k \cdot \sum_{i=0}^{n} i F_{k,n-i} \\
&= n\big(F_{k,n+1} + F_{k,n} - 1\big) - \left(\frac{2}{k} F_{k,n+1} + \frac{-k+2}{k} F_{k,n} - \frac{kn+2}{k} \right) \\
&= \frac{1}{k}\big((kn-2)F_{k,n+1} + (kn+k-2)F_{k,n} + 2\big).
\end{aligned}$$

Proceeding like this, we get the following set of identities:

$$
\begin{aligned}
k \cdot \sum_{i=0}^{n} F_{k,i} &= F_{k,n+1} + F_{k,n} - 1, \\
k \cdot \sum_{i=0}^{n} iF_{k,i} &= \frac{1}{k}\big((kn-2)F_{k,n+1} + (kn+k-2)F_{k,n} + 2\big) \\
&= \frac{1}{k}\big((kn-2)F_{k,n+1} + (k(n+1)-2)F_{k,n} + 2\big), \\
k \cdot \sum_{i=0}^{n} i^2F_{k,i} &= \frac{1}{k^2}\Big(\big(k^2n^2-4kn+8\big)F_{k,n+1} + \big(k^2n^2+2k(k-2)n+k^2-4k+8\big)F_{k,n} - 8\Big) \\
&= \frac{1}{k^2}\Big((k^2n^2-4kn+8)F_{k,n+1} + \big(k^2(n+1)^2-4k(n+1)+8\big)F_{k,n} - 8\Big), \\
k \cdot \sum_{i=0}^{n} i^3F_{k,i} &= \frac{1}{k^3}\Big(\big(k^3n^3-6k^2n^2+24kn-2k^2-48\big)F_{k,n+1} \\
&+ \big(k^3n^3+3k^2(k-2)n^2+3k(k^2-4k+8)n+k^3-8k^2+24k-48\big)F_{k,n} \\
&+ 2k^2+48\Big) \\
&= \frac{1}{k^3}\Big(\big(k^3n^3-6k^2n^2+24kn-2k^2-48\big)F_{k,n+1} \\
&+ \big(k^3(n+1)^3-6k^2(n+1)^2+24k(n+1)-2k^2-48\big)F_{k,n} + 2k^2+48\Big),
\end{aligned} \tag{24}
$$

and so on. As we expected, these equations also follow a pattern. If we define the sums $S_{k,n}^{(p)}$ as

$$
S_{k,n}^{(p)} = \begin{cases} \sum_{i=0}^{n} F_{k,i}, & \text{if } p=0; \\ \sum_{i=0}^{n} i^p F_{k,i}, & \text{if } p \geq 1, \end{cases} \tag{25}
$$

then each equation is of the form

$$
k \cdot S_{k,n}^{(p)} = C_k^{(p)}(n)F_{k,n+1} + C_k^{(p)}(n+1)F_{k,n} - C_k^{(p)}(0),
$$

where $C_k^{(p)}(n)$ is a polynomial in n of degree p. Let us try to investigate the rule of formation of the coefficients of this polynomial, $C_k^{(p)}(n)$. From the last equation in Eq. (24), we identify that

$$
\begin{aligned}
C_k^{(3)}(n) &= \frac{1}{k^3}(k^3n^3-6k^2n^2+24kn-2k^2-48) \\
&= n^3 - \left(\frac{6}{k}\right)n^2 + \left(\frac{24}{k^2}\right)n - \frac{2k^2+48}{k^3} \\
&= 1 \cdot \binom{3}{0}n^3 - \frac{2}{k} \cdot \binom{3}{1}n^2 + \frac{8}{k^2} \cdot \binom{3}{2}n - \frac{2k^2+48}{k^3} \cdot \binom{3}{3},
\end{aligned}
$$

where the numbers 1, $2/k$, $8/k^2$, $(2k^2+48)/k^3$ are the first four terms of the sequence $\left(A_k^{(p)}\right)_{p\geq 0}$. With this in mind, we define the *coefficient polynomial*, $C_k^{(p)}(n)$, in n of degree p as follows:

Definition 5. For all integers $p \geq 0$, we define

$$C_k^{(p)}(n) = \begin{cases} (-1)^p A(p), & \text{if } n = 0; \\ \sum_{r=0}^{p} (-1)^r A_k^{(r)} \binom{p}{r} n^{p-r}, & \text{if } n \neq 0. \end{cases} \tag{26}$$

It should be noted that, for $k > 2$, the polynomial $C_k^{(p)}(n)$ generally doesn't have integer coefficients. From Eqs. (17) and (26), it is clear that $\Phi_k^{(p)}(n) = (-1)^p C_k^{(p)}(-n)$. Consequently, we may rewrite Eq. (15) as

$$k \cdot T_{k,n}^{(p)} = (-1)^p \big(C_k^{(p)}(0) F_{k,n+1} + C_k^{(p)}(1) F_{k,n} - C_k^{(p)}(-n) \big).$$

The central result (7) about the Brousseau sums of the k-Fibonacci numbers can now be established.

Theorem 6. *If $S_{k,n}^{(p)}$ is defined as in Eq. (25), then for all $p \geq 0$, we have*

$$k \cdot S_{k,n}^{(p)} = C_k^{(p)}(n) F_{k,n+1} + C_k^{(p)}(n+1) F_{k,n} - C_k^{(p)}(0), \tag{27}$$

where $C_k^{(p)}(n)$ is the "coefficient polynomial" as defined in Eq. (26).

Proof. If $p = 0$, then the left-hand side of Eq. (27) is

$$k \cdot S_{k,n}^{(p)} = k \cdot \sum_{i=0}^{n} F_{k,i} = F_{k,n+1} + F_{k,n} - 1,$$

and the right-hand side is

$$\begin{aligned} C_k^{(0)}(n) F_{k,n+1} + C_k^{(0)}(n+1) F_{k,n} - C_k^{(0)}(0) &= A_k^{(0)} F_{k,n+1} + A_k^{(0)} F_{k,n} - 1 \\ &= F_{k,n+1} + F_{k,n} - 1. \end{aligned}$$

Thus, Eq. (27) holds for $p = 0$. Now fix $p \geq 1$. Then, using the binomial expansion, we have

$$\begin{aligned} S_{k,n}^{(p)} &= \sum_{i=0}^{n} i^p F_{k,i} \\ &= \sum_{i=0}^{n} (n-i)^p F_{k,n-i} \\ &= \sum_{i=0}^{n} \left(\sum_{r=0}^{p} \binom{p}{r} n^{p-r} (-i)^r \right) F_{k,n-i} \\ &= \sum_{r=0}^{p} (-1)^r \binom{p}{r} n^{p-r} \left(\sum_{i=0}^{n} i^r F_{k,n-i} \right) \\ &= \sum_{r=0}^{p} (-1)^r \binom{p}{r} n^{p-r} T_{k,n}^{(r)}. \end{aligned}$$

Thus,

$$k \cdot S_{k,n}^{(p)} = \sum_{r=0}^{p} (-1)^r \binom{p}{r} n^{p-r} \left(k \cdot T_{k,n}^{(r)} \right). \tag{28}$$

Now, using Theorem 4, we have

$$k \cdot T_{k,n}^{(r)} = A_k^{(r)} F_{k,n+1} + \sum_{j=0}^{r} A_k^{(j)} \binom{r}{j} \big((-1)^{r-j} F_{k,n} - n^{r-j} \big).$$

When we substitute this in Equation (28), we obtain

$$\begin{aligned}
k \cdot S_{k,n}^{(p)} &= \sum_{r=0}^{p} (-1)^r \binom{p}{r} n^{p-r} \left(A_k^{(r)} F_{k,n+1} + \sum_{j=0}^{r} A_k^{(j)} \binom{r}{j} \big((-1)^{r-j} F_{k,n} - n^{r-j} \big) \right) \\
&= \left(\sum_{r=0}^{p} (-1)^r A_k^{(r)} \binom{p}{r} n^{p-r} \right) F_{k,n+1} \\
&+ \sum_{r=0}^{p} \sum_{j=0}^{r} (-1)^r A_k^{(j)} \binom{p}{r} \binom{r}{j} \big((-1)^{r-j} F_{k,n} - n^{r-j} \big) n^{p-r} \\
&= C_k^{(p)}(n) F_{k,n+1} + \sum_{r=0}^{p} \sum_{j=0}^{r} (-1)^r A_k^{(j)} \binom{p}{r} \binom{r}{j} \big((-1)^{r-j} F_{k,n} - n^{r-j} \big) n^{p-r}.
\end{aligned}$$

By switching the order of summation, this becomes

$$\begin{aligned}
k \cdot S_{k,n}^{(p)} &= C_k^{(p)}(n) F_{k,n+1} + \sum_{j=0}^{p} \sum_{r=j}^{p} (-1)^j A_k^{(j)} \binom{p}{r} \binom{r}{j} n^{p-r} F_{k,n} \\
&- \sum_{j=0}^{p} \sum_{r=j}^{p} (-1)^r A_k^{(j)} \binom{p}{r} \binom{r}{j} n^{p-j} \\
&= C_k^{(p)}(n) F_{k,n+1} + \sum_{j=0}^{p} (-1)^j A_k^{(j)} \left(\sum_{r=j}^{p} \binom{p}{r} \binom{r}{j} n^{p-r} \right) F_{k,n} \\
&- \sum_{j=0}^{p} A_k^{(j)} n^{p-j} \left(\sum_{r=j}^{p} \binom{p}{r} \binom{r}{j} (-1)^r \right) \\
&= C_k^{(p)}(n) F_{k,n+1} + \left(\sum_{j=0}^{p} (-1)^j A_k^{(j)} \binom{p}{j} (n+1)^{p-j} \right) F_{k,n} - (-1)^p A_k^{(p)},
\end{aligned}$$

where the last equality follows from the binomial identities

$$\sum_{r=j}^{p} \binom{p}{r} \binom{r}{j} n^{p-r} = \binom{p}{j} (n+1)^{p-j},$$

and

$$\sum_{r=j}^{p} \binom{p}{r} \binom{r}{j} (-1)^r = \begin{cases} 0, & \text{if } j \neq p; \\ (-1)^p, & \text{if } j = p, \end{cases}$$

from Gould's collection [10, Identities 3.118, 3.119, p. 36]. Thus, we conclude that

$$k \cdot S_{k,n}^{(p)} = C_k^{(p)}(n) F_{k,n+1} + C_k^{(p)}(n+1) F_{k,n} - C_k^{(p)}(0).$$

□

Example 7. Setting $p = 2$ in Eq. (27) we get

$$k \cdot \sum_{i=1}^{n} i^2 F_{k,i} = \left(n^2 - \frac{4}{k}n + \frac{8}{k^2}\right) F_{k,n+1} + \left((n+1)^2 - \frac{4}{k}(n+1) + \frac{8}{k^2}\right) F_{k,n} - \frac{8}{k^2}.$$

In particular, when $k = 11$ this becomes

$$11 \cdot \sum_{i=1}^{n} i^2 F_{11,i} = \left(n^2 - \frac{4}{11}n + \frac{8}{121}\right) F_{11,n+1} + \left((n+1)^2 - \frac{4}{11}(n+1) + \frac{8}{121}\right) F_{11,n} - \frac{8}{121}.$$

Since $F_{11,i} = F_{5i}/F_5 = F_{5i}/5$ (see [9]), we obtain

$$11 \cdot \sum_{i=1}^{n} i^2 F_{5i} = \left(n^2 - \frac{4}{11}n + \frac{8}{121}\right) F_{5n+5} + \left((n+1)^2 - \frac{4}{11}(n+1) + \frac{8}{121}\right) F_{5n} - \frac{5 \cdot 8}{121},$$

which gives the identity about the Brousseau sums of the sequence $\left(F_{5i}\right)_{i \geq 1}$.

Example 8. Setting $p = 2$ and $k = 14$ in Eq. (27) we get

$$14 \cdot \sum_{i=1}^{n} i^2 F_{14,i} = \left(n^2 - \frac{2}{7}n + \frac{2}{49}\right) F_{14,n+1} + \left((n+1)^2 - \frac{2}{7}(n+1) + \frac{2}{49}\right) F_{14,n} - \frac{2}{49}.$$

Since $F_{14,i} = P_{3i}/P_3 = P_{3i}/5$ (see [9]), we obtain

$$14 \cdot \sum_{i=1}^{n} i^2 P_{3i} = \left(n^2 - \frac{2}{7}n + \frac{2}{49}\right) P_{3n+3} + \left((n+1)^2 - \frac{2}{7}(n+1) + \frac{2}{49}\right) P_{3n} - \frac{5 \cdot 2}{49},$$

which gives the identity about the Brousseau sums of the sequence $\left(P_{3i}\right)_{i \geq 1}$.

Examples 7 and 8 suggest two interesting identities Eqs. (29) and (31). Eq. (29) is about the Brousseau sums of every m^{th} Fibonacci number, and Eq. (31) is that of every m^{th} Pell number, when m is odd.

Corollary 9. *Let $m \geq 1$ be an odd integer. Then for all integers $p \geq 0$, the following identity holds:*

$$L_m \cdot \sum_{i=1}^{n} i^p F_{mi} = C_{L_m}^{(p)}(n) F_{m(n+1)} + C_{L_m}^{(p)}(n+1) F_{mn} - C_{L_m}^{(p)}(0) F_m, \tag{29}$$

where L_m is the m^{th} Lucas number.

Proof. Setting $k = L_m$ in Eq. (27) yields

$$L_m \cdot \sum_{i=1}^{n} i^p F_{L_m,i} = C_{L_m}^{(p)}(n) F_{L_m,n+1} + C_{L_m}^{(p)}(n+1) F_{L_m,n} - C_{L_m}^{(p)}(0). \tag{30}$$

If m is odd, then we have (see [9])

$$F_{L_m,i} = \frac{F_{mi}}{F_m}.$$

Applying this in Eq. (30) and multiplying through by F_m, we get Eq. (29). □

Corollary 10. *Let $m \geq 1$ be an odd number. Then for all integers $p \geq 0$, the following identity holds:*

$$Q_m \cdot \sum_{i=1}^{n} i^p P_{mi} = C_{Q_m}^{(p)}(n) P_{m(n+1)} + C_{Q_m}^{(p)}(n+1) P_{mn} - C_{Q_m}^{(p)}(0) P_m, \tag{31}$$

where Q_m is the m^{th} Pell-Lucas number.

Proof. The proof is similar to the proof of Corollary (9) by using the fact that (see [9])

$$F_{Q_m,i} = \frac{P_{mi}}{P_m},$$

when m is odd. □

5 Shifted Brousseau sums

In this section, we find the formula for the shifted Brousseau sums

$$\sum_{i=1}^{n} i^p F_{k,m+i},$$

for all integers $m, p \geq 0$. For example, if we take $p = 1$, then

$$\begin{aligned}
k \cdot \sum_{i=1}^{n} iF_{k,m+i} &= k \cdot \sum_{i=m+1}^{m+n} (i-m)F_{k,i} \\
&= k \cdot \sum_{i=m+1}^{m+n} iF_{k,i} - mk \cdot \sum_{i=m+1}^{m+n} F_{k,i} \\
&= k\left(\sum_{i=0}^{m+n} iF_{k,i} - \sum_{i=0}^{m} iF_{k,i}\right) - mk\left(\sum_{i=0}^{m+n} F_{k,i} - \sum_{i=0}^{m} F_{k,i}\right).
\end{aligned}$$

Now, using the first two identities in Eq. (24), we have

$$\begin{aligned}
k \cdot \sum_{i=1}^{n} iF_{k,m+i} &= \frac{1}{k}\big((k(m+n)-2)F_{k,m+n+1} + (k(m+n+1)-2)F_{k,m+n} - (km-2)F_{k,m+1} \\
&\quad - (k(m+1)-2)F_{k,m}\big) - m\big(F_{k,m+n+1} + F_{k,m+n} - F_{k,m+1} - F_{k,m}\big) \\
&= \frac{1}{k}\big((kn-2)F_{k,m+n+1} + (k(n+1)-2)F_{k,m+n} + 2F_{k,m+1} + (-k+2)F_{k,m}\big).
\end{aligned}$$

We generalize this identity for all integers $p \geq 0$ in the next theorem.

Theorem 11. *For all integers $m, p \geq 0$, we have*

$$k \cdot \sum_{i=1}^{n} i^p F_{k,m+i} = C_k^{(p)}(n)F_{k,m+n+1} + C_k^{(p)}(n+1)F_{k,m+n} - C_k^{(p)}(0)F_{k,m+1} - C_k^{(p)}(1)F_{k,m}. \quad (32)$$

Proof. Since $F_{k,0} = 0$ and $F_{k,1} = 1$, the case $m = 0$ follows from Theorem 6. Now fix $m \geq 1$. Then, using the binomial expansion, we have

$$\begin{aligned}
\sum_{i=1}^{n} i^p F_{k,m+i} &= \sum_{i=m+1}^{m+n} (i-m)^p F_{k,i} \\
&= \sum_{i=m+1}^{m+n} \sum_{j=0}^{p} \binom{p}{j} i^{p-j}(-m)^j F_{k,i} \\
&= \sum_{j=0}^{p} \binom{p}{j} (-m)^j \sum_{i=m+1}^{m+n} i^{p-j} F_{k,i} \\
&= \sum_{j=0}^{p} \binom{p}{j} (-m)^j \big(S_{k,m+n}^{(p-j)} - S_{k,m}^{(p-j)}\big).
\end{aligned}$$

Thus,

$$k \cdot \sum_{i=1}^{n} i^p F_{k,m+i} = \sum_{j=0}^{p} \binom{p}{j} (-m)^j \big(k \cdot S_{k,m+n}^{(p-j)} - k \cdot S_{k,m}^{(p-j)}\big).$$

Now, applying Theorem 6, this becomes

$$
\begin{aligned}
k \cdot \sum_{i=1}^{n} i^p F_{k,m+i} = \sum_{j=0}^{p} \binom{p}{j} (-m)^j \Big(& C_k^{(p-j)}(m+n) F_{k,m+n+1} + C_k^{(p-j)}(m+n+1) F_{k,m+n} \\
& - C_k^{(p-j)}(m) F_{k,m+1} - C_k^{(p-j)}(m+1) F_{k,m} \Big)
\end{aligned}
\tag{33}
$$

Consider

$$
\sum_{j=0}^{p} \binom{p}{j} (-m)^j C_k^{(p-j)}(m+n) = \sum_{j=0}^{p} \binom{p}{j} (-m)^j \sum_{r=0}^{p-j} (-1)^r A_k^{(r)} \binom{p-j}{r} (m+n)^{p-j-r}.
$$

By switching the order of summation, this becomes

$$
\sum_{j=0}^{p} \binom{p}{j} (-m)^j C_k^{(p-j)}(m+n) = \sum_{r=0}^{p} \sum_{j=0}^{p-r} (-1)^r A_k^{(r)} \binom{p}{j} \binom{p-j}{r} (m+n)^{p-j-r} (-m)^j. \tag{34}
$$

Next, we use the binomial identity [2, Identity 134, p. 67] to get

$$
\binom{p}{j} \binom{p-j}{r} = \binom{p}{p-j} \binom{p-j}{r} = \binom{p}{r} \binom{p-r}{j}.
$$

Substituting this in Eq. (34), we obtain

$$
\begin{aligned}
\sum_{j=0}^{p} \binom{p}{j} (-m)^j C_k^{(p-j)}(m+n) &= \sum_{r=0}^{p} (-1)^r A_k^{(r)} \binom{p}{r} \left(\sum_{j=0}^{p-r} \binom{p-r}{j} (m+n)^{p-r-j} (-m)^j \right) \\
&= \sum_{r=0}^{p} (-1)^r A_k^{(r)} \binom{p}{r} n^{p-r} \\
&= C_k^{(p)}(n).
\end{aligned}
\tag{35}
$$

Similarly, we can show that

$$
\sum_{j=0}^{p} \binom{p}{j} (-m)^j C_k^{(p-j)}(m+n+1) = C_k^{(p)}(n+1), \tag{36}
$$

$$
\sum_{j=0}^{p} \binom{p}{j} (-m)^j C_k^{(p-j)}(m) = C_k^{(p)}(0), \tag{37}
$$

and

$$
\sum_{j=0}^{p} \binom{p}{j} (-m)^j C_k^{(p-j)}(m+1) = C_k^{(p)}(1). \tag{38}
$$

Thus, Eq. (32) follows by substituting Eq. (35) through Eq. (38) in Eq. (33). □

6 Conclusion

While all the results presented above assume that k is a positive integer, there is no reason not to extend them to nonzero real numbers as well. The only drawback is that the numbers $F_{k,n}$ are not necessarily integers. The k-Fibonacci numbers are just the Fibonacci polynomials $F_n(x)$ (see [3]) calculated at $x = k$. Hence, we strongly believe that all the above results are still valid if we allow non-integer values of k. For example, we can have the identity

$$\sqrt{2} \cdot \sum_{i=1}^{n} i^2 F_i\big(\sqrt{2}\big) = \big(n^2 - 2\sqrt{2}n + 4\big) F_{n+1}\big(\sqrt{2}\big) + \big((n+1)^2 - 2\sqrt{2}(n+1) + 4\big) F_n\big(\sqrt{2}\big) - 4.$$

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