A UNIVERSAL ALGORITHM FOR PHOTOGRAMMETRIC COMPUTATIONS

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1. GENERAL DESCRIPTION

The flexibility of photogrammetric equipment systems and techniques has grown by the influence of space travel and computer technology. In addition to photogrammetric cameras use is now made of non-metric or partially metric cameras, opto-mechanical and opto-electronic strip cameras for data acquisition. The portions of the earth's surface covered from space are now - depending of the recording system - so large that distortions occuring in special map imagery have to be taken into consideration in restitution. Both, in photogrammetric software packages (e.g. for aerial triangulation) and in analytical plotters the resulting requirements cannot all be satisfied with the normally implemented mathematical model of central perspective.

In order to overcome this state an algorithm is suggested, which universally applies to all kinds of picture and arbitrary object coordinate systems and in which the differences of the individual mathematical models can be allocated to the process of image orientation. The description of the algorithm is given for the computation of image coordinates from object coordinates, which is the basis of all analytical plotters. Starting point of the universal algorithm is a general formulation of the image equation, with the relationships between object and image coordinates being separated into several transformations (Fig. 1) and treated by the theorem of Taylor.

In Fig. 1 the transformation $x' = F(x)$ applies to the mathematical model of the recording process, $x = U(\mathbf{x})$ to the relations between two object coordinate systems, and $x'_K = G(x')$ to the influence of systematic errors on the image coordinates.

According to the theorem of Taylor each function $f = f(x,y,z)$ beeing (r+1)-times continuously differentiable can be developed in the environment of a point $P_O(x_O, y_O, z_O)$. Hence, an algorithm based on Taylor's formula operates as follows. The object space is subdivided into spatial segments. The centre coordinates, the appertaining image coordinates, the appertaining values of the partial differentials and of the remainder terms are stored for each segment. The transition from one segment to a neighbouring segment is always associated with a change of the entire parameter set.

If the theorem of Taylor is applied to

$$x'_i = f_i(x, y, z) \qquad (1)$$

with r = 1 and the index i characterizing a special image coordinate is dispensed with, one obtains the following relation for the image coordinates

$$x' = x'_o + F_x \cdot dx + R_{x'} \tag{2}$$

$$dx^T = (dx \quad dy \quad dz) = (x-x_o \quad y-y_o \quad z-z_o)$$

$$F_x = (f_x \quad f_y \quad f_z)$$

F_x is the matrix of the partial derivatives.
The Lagrange remainder term reads

$$R_{x'} = \tfrac{1}{2} dx^T \cdot F_{xx} \cdot dx \qquad F_{xx} = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix}$$

$$f_{xx} = f_{xx}(x_o + \delta dx, y_o + \delta dy, z_o + \delta dz), \ldots \delta \in (0,1)$$

are the 2nd derivatives of f.

The problem of the calculation of the remainder term $R_{x'}$ or of the matrix F_{xx} lies in the determination of the factor δ, which in the general case $\delta = \delta(x,y,z)$ is dependent on x_o, y_o, z_o as well as on dx, dy, dz, i.e. it changes from point to point. To investigate possibilities of the exact calculation or of the improvement of the remainder term, $R_{x'}^{Lagr}$ is determined also by inversion of (2) and compared with $R_{x'}$.

If the evalution is not to be made in the coordinate system (x,y,z), but in a coordinate system $(\bar{x},\bar{y},\bar{z})$, for which the following relation holds true

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u(\bar{x},\bar{y},\bar{z}) \\ v(\bar{x},\bar{y},\bar{z}) \\ w(\bar{x},\bar{y},\bar{z}) \end{bmatrix} = U(\bar{x}), \tag{3}$$

Taylor's formula must be rewritten as follows:

$$x' = x'_o + F_{\bar{x}} \cdot d\bar{x} + \bar{R}_{x'} \qquad \bar{R}_{x'} = \tfrac{1}{2} \cdot d\bar{x}^T \cdot F_{\bar{x}\bar{x}} \cdot d\bar{x} \qquad (4),\ (5)$$

In the calculation of the partial derivatives in (4) and (5) the substitution of (3) in (2) is understood to be a change of the variable and the chain rule is applied in the further process. In matrix notation it holds for the first derivatives

$$F_{\bar{x}} = (f_{\bar{x}} \quad f_{\bar{y}} \quad f_{\bar{z}}) = F_x \cdot U_{\bar{x}}, \qquad U_{\bar{x}} = \begin{bmatrix} u_{\bar{x}} & u_{\bar{y}} & u_{\bar{z}} \\ v_{\bar{x}} & v_{\bar{y}} & v_{\bar{z}} \\ w_{\bar{x}} & w_{\bar{y}} & w_{\bar{z}} \end{bmatrix} \tag{6}$$

Also the second derivatives can be combined in a matrix $F_{\bar{x}\bar{x}}$

$$F_{\bar{x}\bar{x}} = U_{\bar{x}}^{T} \cdot F_{xx} \cdot U_{\bar{x}} + \bar{F}_{x} \cdot U_{\bar{x}\bar{x}} \tag{7}$$

$$\bar{F}_{x} = \begin{bmatrix} F_x & O & O \\ O & F_x & O \\ O & O & F_x \end{bmatrix} \qquad O = (1,3) \text{ null-matrix}$$

$$U_{\bar{x}\bar{x}}^{T} = \begin{bmatrix} u_{\bar{x}\bar{x}} & v_{\bar{x}\bar{x}} & w_{\bar{x}\bar{x}} & u_{\bar{x}\bar{y}} & v_{\bar{x}\bar{y}} & w_{\bar{x}\bar{y}} & u_{\bar{x}\bar{z}} & v_{\bar{x}\bar{z}} & w_{\bar{x}\bar{z}} \\ u_{\bar{x}\bar{y}} & v_{\bar{x}\bar{y}} & w_{\bar{x}\bar{y}} & u_{\bar{y}\bar{y}} & v_{\bar{y}\bar{y}} & w_{\bar{y}\bar{y}} & u_{\bar{y}\bar{z}} & v_{\bar{y}\bar{z}} & w_{\bar{y}\bar{z}} \\ u_{\bar{x}\bar{z}} & v_{\bar{x}\bar{z}} & w_{\bar{x}\bar{z}} & u_{\bar{y}\bar{z}} & v_{\bar{y}\bar{z}} & w_{\bar{y}\bar{z}} & u_{\bar{z}\bar{z}} & v_{\bar{z}\bar{z}} & w_{\bar{z}\bar{z}} \end{bmatrix}$$

Then the evaluation in a coordinate system $(\bar{x}, \bar{y}, \bar{z})$ can be realized in such a way that for each segment the calculation of the elements of the matrices $U_{\bar{x}}$ and $U_{\bar{x}\bar{x}}$ is additionally performed and the matrix multiplications shown in (6) and (7) are carried out.

The use of $U(\bar{x})$ presumes that the relations existing between the object coordinate systems (x, y, z) and $(\bar{x}, \bar{y}, \bar{z})$ are explicitly presented and that all parameters occurring in (3) are known.

If systematic errors are still contained in the images, the final image coordinates must then be calculated by

$$x'_k = g(x', y')$$

By application of Taylor's formula one obtains

$$x'_k = x'_{ko} + G_{x'} \cdot dx' + R_{xk'} \tag{8}$$

with $G_{x'} = (g_{x'} \quad g_{y'})$, $dx' = (dx' \quad dy')^T = \left[(x'-x'_o) \quad (y'-y'_o) \right]^T$

$$R_{xk'} = ½\, dx'^{T} \cdot G_{x'x'} \cdot dx' ,$$

By substitution of (4) it follows

$$x'_k = x'_{ko} + G_{x'} \begin{bmatrix} F_{1\bar{x}} \cdot d\bar{x} + \bar{R}_{x'} \\ F_{2\bar{x}} \cdot d\bar{x} + \bar{R}_{y'} \end{bmatrix} = x'_{ko} + G_{x'} \begin{bmatrix} F_{1\bar{x}} \\ F_{2\bar{x}} \end{bmatrix} \cdot d\bar{x} + \begin{bmatrix} \bar{R}_{x'} \\ \bar{R}_{y'} \end{bmatrix} \tag{9}$$

It depends on the amount of the image errors to be corrected if the remainder term in (8) and the multiplication of $R_{x'}$ and $R_{y'}$ by $G_{x'}$ can generally be neglected or not.

(4) and (9) are valid independent of whether the images were taken with a photogrammetric camera, strip camera, panorama camera or a scanner. If the remainder terms cannot rigorously be calculated, then the remaining errors have to be ascertained and must be taken into account in the specification of the maximally admissible dx, dy, dz (i.e. of the environment of the local zero points $P_O(x_O, y_O, z_O)$, in which Taylor's formula may be used). Thus, the differences between the individual image types are part of a preceding program, in which the local zero points P_{Oj} within the model are fixed and the appertaining local zero points in the images as well as the partial derivatives of the different matrices are calculated.

The use of 2 image coordinate systems as well as of 2 object coordinate system renders it possible to expediently split the mathematical relations between image and object coordinates. Expedient allocations may for example be:

Aerial photogrammetry

U(X) : Earth curvature, refraction, special map projection
F(x) : Central perspective
G(x'): Lens distortion, film deformation

Terrestrial photogrammetry

U(X) : Special object coordinate systems (e.g. cylindrical coordinates), two-media photogrammetry
F(x) : Central perspective
G(x'): Lens distortion, film deformation

Evalution of scenes taken by opto-mechanical scanners

U(X) : -
F(x) : Polynomial transformation
G(x'): -

Evalution of scenes taken by pushbroom scanners from space

U(X) : Special map projection, earth curvature perpendicular to the flying direction, refraction
F(x) : Central perspective inclusive of the temporal change of of orientation elements, earth curvature in flying direction
G(x'): -

Some components of this survey are described in the following Sections.

2. THE MATHEMATICAL MODEL OF THE TAKING PROCESS

The mathematical model of the taking process with photogrammetric cameras is the central perspective, which is described whith the known collinearity equations

$$\begin{bmatrix} x' \\ y' \\ -c_k \end{bmatrix} = \frac{1}{\lambda'} A^T \begin{bmatrix} x-x_{o1} \\ y-y_{o1} \\ z-z_{o1} \end{bmatrix}$$

They were investigated in Mark, 1986 by the principles described in Section 1 and led to the <u>rigorous</u> solution

$$x' = x'_o + \frac{\zeta_o}{\zeta_o + d\zeta} \cdot F_{1x} \cdot dx \qquad y' = y'_o + \frac{\zeta_o}{\zeta_o + d\zeta} \cdot F_{2x} \cdot dx \tag{10}$$

$$F_{1x} = -\frac{1}{\zeta_o}(c_k \quad 0 \quad x'_o) \cdot A^T, \qquad F_{2x} = -\frac{1}{\zeta_o}(0 \quad c_k \quad y'_o) \cdot A^T$$

$$\zeta_o = a_{13}(x_o - x_{o1}) + a_{23}(y_o - y_{01}) + a_{33}(z_o - z_{o1})$$

$$d\zeta = a_{13}dx + a_{23}dy + a_{33}dz$$

A second example refers to the evalution of scanner scenes, for which the starting equations were given by Konecny, 1976. These equations must however be modified depending on the type of the scanner. For demonstration, reference is made in the following to the French satellite SPOT, for which it is assumed on the basis of data given by Guichard, 1983 and Toutin, 1986 that the scanner is very well stabilized and hence

- the motions of the projection centre of the scanner during the scanning process of a scene are performed with constant speed in an undisturbed orbit which is approximated by the curvature circle
- the angle orientation in space is a linear function of time or of the distance covered.

Without dealing with the derivations in detail, merely the result is given here. It reads for x' (see equation (11)).

The coefficients a_1 to d_9 are calculated from the rotation matrix for the central pixel of the scene, the linear changes of the orientation elements within the scene, and the influence of the earth's rotation. In (11) the influence of the earth's curvature in flying direction has been taken into consideration. The treatment according to section 1 yields for x' to equation (12)

The elements (x1) to (x61) are calculated from the coefficients a_1 to d_9 and the coordinate differences $(x_o - x_{o1}, (y_o - y_{o1}), (z_o - z_{o1})$.

$$x' = -c_k \cdot \frac{a_1(x_i - x_{oi}) + a_2(y_i - y_{oi}) + a_3(z_i - z_{oi}) + a_4(x_i - x_{oi})(y_i - y_{oi}) + a_5(y_i - y_{oi})^2 + a_6(y_i - y_{oi})(z_i - z_{oi}) + a_7(x_i - x_{oi})(y_i - y_{oi})^2 + a_8(y_i - y_{oi})^3 + a_9(y_i - y_{oi})^2(z_i - z_{oi})}{b_1(x_i - x_{oi}) + b_2(y_i - y_{oi}) + b_3(z_i - z_{oi}) + b_4(x_i - x_{oi})(y_i - y_{oi}) + b_5(y_i - y_{oi})^2 + b_6(y_i - y_{oi})(z_i - z_{oi}) + b_7(x_i - x_{oi})(y_i - y_{oi})^2 + b_8(y_i - y_{oi})^3 + b_9(y_i - y_{oi})^2(z_i - z_{oi})} \quad (11)$$

$$x' = x'_o + \frac{\xi x_o}{\xi x_o + d\xi x} \left\| F_{1x} dx - \frac{1}{2\xi x_o} dx^T \cdot dAx \cdot dx \right\| \quad (12)$$

$$F_{1x} = -\frac{1}{\xi x} \cdot (c_k \quad x'_o) \begin{Vmatrix} (x1) & (x2) & (x3) \\ (x4) & (x5) & (x6) \end{Vmatrix}$$

$$dAx = \begin{Vmatrix} 0 & (x11)c_k + (x41)x'_o & 0 \\ (x11)c_k + (x41)x'_o & (x21)c_k + (x51)x'_o & (x31)c_k + (x61)x'_o \\ 0 & (x31)c_k + (x61)x'_o & 0 \end{Vmatrix}$$

3. THE EXTENSION OF THE TAKING PROCESS TO THE OBJECT COORDINATE SYSTEM $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ AND THE IMAGE COORDINATE SYSTEM $(x'_k, \ldots)$

For central-perspective images the combination of (4) to (7) with (10) leads to the equations

$$x' = x'_o + \frac{\bar{\xi}_o}{\bar{\xi}_o + d\bar{\xi}} (K_1 \cdot d\mathbf{X} + \tfrac{1}{2} d\mathbf{X}^T \cdot R_1 \cdot d\mathbf{X}) \quad (13)$$

$$K_1 = F_{1x} \cdot U_{\mathbf{X}} \quad , \quad R_1 = \bar{F}_{1x} \cdot U_{\mathbf{XX}}$$

For SPOT scenes one obtains by combination of (4) to (7) with (12)

$$x' = x'_o + \frac{\bar{\xi} x_o}{\bar{\xi} x_o + d\bar{\xi} x} \left\| K_1 \cdot d\mathbf{X} + \tfrac{1}{2} \cdot d\mathbf{X}^T \cdot R_1 \cdot d\mathbf{X} \right\| \quad (14)$$

$$K_1 = F_{1x} \cdot U_{\mathbf{X}} \qquad R_1 = \bar{F}_{1x} \cdot U_{\mathbf{XX}} - \frac{1}{\bar{\xi} x_o} U^T_{\mathbf{X}} \cdot dAx \cdot U_{\mathbf{X}} \quad (15)$$

Expanding (13) to include the consideration of systematic image errors leads according to (9) again to (13) with

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} := \begin{bmatrix} G_{1x'} \\ G_{2x'} \end{bmatrix} \cdot \begin{bmatrix} F_{1x} \\ F_{2x} \end{bmatrix} \cdot U_{\bar{x}}$$

This expansion shall not be further considered here.

The comparison of (13) with (14) shows that (14) is the wanted universal algorithm.

4. EXAMPLES FOR $(\bar{x}, \bar{y}, \bar{z})$ OBJECT COORDINATE SYSTEM

If a restitution is to be made in an $(\bar{x}, \bar{y}, \bar{z})$ coordinate system rather than in an (x, y, z) one, the following prerequisites have to be established according to sect. 1. First, establish $x = U(\bar{x})$ (3). From this, form the matrices of the partial derivatives $U_{\bar{x}}$ and $U_{\bar{x}\bar{x}}$. If (3) is used in the form of approximative solutions or series expansions, it is necessary to ensure that the replacement function represents as correctly as possible not only the initial function but also its partial derivatives, because this greatly bears on the the segment dimensions. Finally, compute for every segment the amounts of the partial derivatives and the matrices K and R, according to (15).

In the following examples, only the relations $x = U(\bar{x})$ are given for lack of space.

4.1. Linear dependence

As an example from architectural photogrammetry, Mark, 1986 reported on the restitution in an object coordinate system that was tilted relative to the taking base or to the control coordinate system.

4.2. Allowance for earth curvature and refraction

Earth curvature may be allowed for both with photo and model coordinates. Correction of the model coordinates is a rigid solution, whereas the correction of photo coordinates is approximative for all nadir distances deviating from zero. Both in aerial and satellite photographs greater photo tilts must be expected, so that the rigid solution should be preferred.

The horizontal plane of the photogrammetric (x, y, z) coordinate system touches the datum surface of the normal height system, which may here be assumed to be a spherical cap, at point A. To obtain symmetric conditions with regard to the influence of earth curvature in the left and right photographs, we select

$$x_A = x_{o1} + ½ \cdot bx \qquad y_A = y_{o1} + ½ \cdot by$$

The curvature of the imaging ray due to refraction has the effect that object point, projection centre and image point do not be on a straight line. The error thus produced can be allowed for in the object coordinate system.

If a mean terrain height is taken for a model, refraction is compensated with sufficient accuracy by $\Delta\alpha = k_{Refr.} \cdot \tan\alpha$, with $k_{Refr.}$ = constant. Thus, we can write for the left photo for (3)

$$x_1 - x_{o1} = \bar{X} \quad , \qquad y_1 - y_{o1} = \bar{Y}$$

$$z_1 = \bar{Z} - \frac{1}{2R}[(\bar{X}-\tfrac{1}{2}bx)^2 + (\bar{Y}-\tfrac{1}{2}by)^2] + (1 + \frac{\bar{X}^2 + \bar{Y}^2}{(\bar{Z}-z_{o1})^2})(z-z_{o1}) \cdot k_{Refr}$$

4.3. Two-media photogrammetry

Two-media photogrammetry is a special case of multimedia photogrammetry. It is concerned with measurements in photographs taken through two media of different density separated by a (plane) interface (Höhle, 1971). In many cases - in hydroengineering model experiments or in shallow water surveys - the interface is horizontal, which further simplifies the solution of the task.

Following Höhle, 1971, the (x, y, z) coordinate system is placed into the interface G . In this coordinate system, that part of the imaging ray which belongs to the medium of refractive index n_α is described. The other part of the imaging ray lying in the medium of the refractive index n_β is described in the $(\bar{x}, \bar{y}, \bar{z})$ coordinate system. The problem consists in establishing a relationsship between the two coordinate systems. The derivation leads to

$$f(r) = \bar{r} - r(1 + \frac{\bar{Z}}{z_{o1}NK}) = 0 \ , \quad K = (1 + \frac{N^2-1}{N^2} \cdot \frac{r^2}{z_{o1}^2}) \ , \quad N = \frac{n_\beta}{n_\alpha} \qquad (16)$$

which is developed into a Taylor series.

From the first two summands there results Newton's well-known approximation formula, which is substituted for the terms with the second and higher derivatives and solved with regard to r:

$$r = r_o + \frac{z_{o1}NK(\bar{r}-r_o)-r_o\bar{Z}}{z_{o1}NK^3 + \bar{Z}} \cdot K^2 \cdot \left[1 + \frac{3(N^2-1)r_o\bar{Z}[z_{o1}NK(\bar{r}-r_o)-r_o\bar{Z}]}{2N^2 z_{o1}^2 \cdot (z_{o1}NK^3 + \bar{Z})^2}\right] \qquad (17)$$

In K set $r = r_o$.

With (17) it is possible (cf. Masry &Konecny, 1970) by the introduction of separate space points P_G for the left and right photographs, to establish the wanted functions u, v, w.

$$x - x_{o1} = \frac{r}{\bar{r}} \cdot \bar{X} \ , \quad y - y_{o1} = \frac{r}{\bar{r}} \cdot \bar{Y} \ , \quad z = 0$$

4.4. Restitution of cylindrical coordinates

Cylinders are a basic form in building construction and industry. They not only occur in towers or tanks but also are fundamental to vaults. For the optimum preparation for reconstruction jobs, cylindrical surfaces often have to be developed.

The connection between the (x, y, z) and (x̄, ȳ, z̄) coordinate systems is given by Mark, 1986.

5. SUMMARY AND OUTLOOK

As explained in section 1 to 4, it is possible, by means of Taylor's formula and a special form of the remainder, to derive an algorithm which is universally applicable to all mathematical models of the taking geometry, to the correction of photo and model coordinates for systematic errors, and to the introduction of non-cartesian model (object) coordinates, as demonstrated in section 4 by several examples from terrestrial and aerial photogrammetry.

In addition to its universal applicability, the algorithm has an other salient feature - it is a rigid solution for the restitution of frame photographs in a cartesian model coordinate system. Given the high quality demands placed on photogrammetric restitution, this is a remarkable advantage for the majority of restitution assignments, which places the proposed algorithm on the same level, in terms of usefulness and rank, with the collinearity equations of centrally perspective projection.

The proposed algorithm has many potential uses.

Firstly, it suggests itself for use with digitally controlled photogrammetric restitution machines such as analytical stereoplotters or orthoprinters, and for analytical single-photo restitution.

Moreover, it is possible to employ the proposed algorithm in systems based an the principles of digital image processing. The analysis of image section common in that field even accomodates the segmentalization of the model space, because it requires less frequent changes of parameter sets than the rather object-related compilation in the classical photogrammetric restitution instruments. Thus it appears feasible that parameter sets, rather than being stored in toto, be computed during the time that is needed anyhow for constructing the image on the video screen.

The proposed algorithm may just as well be used as a basis for photogrammetric computing programs such as for aerotriangulation. These programs, which so far were only applicable for a limited range of applications, now become open for implementing any camera-to-object geometry and any object coordinate system, without the need of interfering with fundamental program structures.

Thus, in the restitution of photographs and the processing of digital data collected with opto-electronic systems, the universal algorithm proposed is an alternative to the use of special mathematical models.

6. REFERENCES

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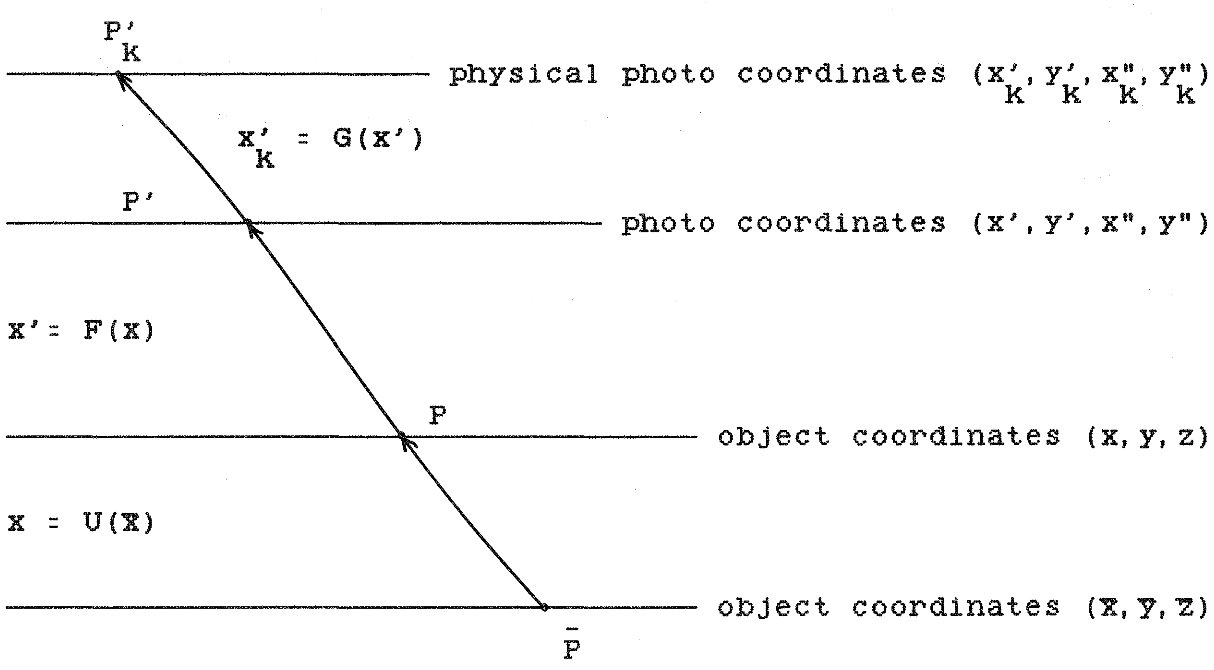

Fig. 1. Transformation of object coordinates into photo coordinates